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ALL INTERVAL GREEDOIDS ON AT MOST 5 ELEMENTS

Dragan M. Acketa
Institute of Mathematics, 21000 Novi Sad,
Dr Ilije Djuričića 4, Yugoslavia

ABSTRACT

An algorithm for checking intervality of a greedoid is described. We apply the corresponding computer program to the list of all non-isomorphic greedoids on at most 5 elements, which was obtained in [1]. Greedoid denotations in the two papers are in accordance.

PRELIMINARIES

A *greedoid* ([2]) on a finite ground-set E is a pair (E,F), where $F \subseteq 2^E$ (family of *feasible* sets) satisfies

(i) $\emptyset \in F$

(ii) $\forall X \in F, \exists x \in X$ so that $X-x \in F$

(iii) $X,Y \in F$, $|X| = |Y|+1$, $\exists x \in X-Y$ so that $Y \cup x \in F$.

Two greedoids (E,F1) and (E,F2) are *isomorphic* if there is a bijection between their ground-sets, which maps the feasible sets of one greedoid onto the feasible sets of the other.

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An important class of greedoids fulfils the following "intervality condition":

A greedoid (E,F) is *interval* if for all $A \subseteq B \subseteq C \subseteq E$ and $x \in E-C$, such that $A, B, C, A \cup x, C \cup x \in F$, it also holds that $B \cup x \in F$.

Let "mc" denote the maximal cardinality of a feasible set. An *n-set* is a set of cardinality n. Sets are denoted without brackets and commas.

A greedoid G2=(E,F2) is said to be *erected* from a greedoid G1=(E,F1) if $F1 \subset F2$ and the minimal cardinality of a feasible set in F2-F1 is strictly greater than the maximal cardinality of a feasible set in F1.

Given $|E|=n$, it is almost obvious ([1]) that the greedoids with mc=n-1 and mc=n can be coupled together so that two greedoids in the same couple differ solely in feasibility of E.

LEMMA. *Let* G1 *and* G2 *be two greedoids in the same couple, so that* mc=n-1 *with* G1 *and* mc=n *with* G2. *If* G1 *is non-interval, then* G2 *is non-interval, too.*

PROOF. Both greedoids may use the same intervality counterexample.□

Critical 6-tuples

A *critical* 6- tuple of subsets of E is a 6-tuple of the form $(A, B, C, A \cup x, B \cup x, C \cup x)$, where $A \subset B \subset C \subset E$ and $x \in E-C$. A critical 6-tuple is *non-interval* if it gives a counterexample for greedoid intervality (that is, if the fifth set $B \cup x$ is the only non-feasible set among the six sets).

Let a,b,c,n denote the cardinalities of the sets A,B,C,E respectively. Given the numbers a,b,c,n, it is obvious that the number of the critical 6-tuples is equal to

$$f(a,b,c,n) \overset{\text{def}}{=} \binom{n}{a}\binom{n-a}{b-a}\binom{n-b}{c-b}\binom{n-c}{1}$$

The total number of critical 6-tuples for a fixed n is equal to

$$g(n) \overset{\text{def}}{=} \sum_{0 \leq a < b < c < n} f(a,b,c,n)$$

(this sum has $\binom{n}{3}$ summands).

The values of $g(n)$ for n between 1 and 7 are equal to 0, 0, 6, 72, 550, 3420, 18194 respectively.

Critical 6-tuples constitute a basis for checking intervality of greedoids. Thus each interval greedoid on 5 elements should pass 550 tests before we claim its intervality. On the other hand, the first non-interval critical 6-tuple proves that the greedoid is non-interval, too.

Coding

Let $E=\{1,\ldots,n\}$. A critical 6-tuple $(A,B,C,A\cup x,B\cup x,C\cup x)$ on E is represented by a word of length k $(7 \leq k \leq n+4)$, which contains exactly four zeros, while the remaining letters are distinct elements from E. The four zeros are located on the positions with the ordinal numbers $i,j,k-2,k$ respectively, where $0<i<i+1<j<k-3$. The non-zero elements before each of the four zeros determine (i.e., belong to) the sets A, $B-A$, $C-B$, $\{x\}$, in this order.

EXAMPLE. The word 0123045060 determines the 6-tuple $(\emptyset,123,12345,6,1236,123456)$.

In order to get a suitable greedoid representation, we define a natural bijection nb, which maps the subsets of E onto the set $\{0,1,\ldots,2^n-1\}$: If $X \subseteq E$, then let first "x" denote the n-dimensional 0-1 vector, which satisfies $x(i)=1$ iff $i \in X$, $1 \leq i \leq n$. Then $nb(X)$ is the integer with binary expansion x. Finally, a family F of feasible subsets of E is represented by the (2^n)-dimensional vector 0-1 v satisfying $v(nb(X)) = 1$ iff $X \in F$.

EXAMPLE. The family F={0,1,12,13} on E={1,2,3} is represented by v=(10001110) (the sets of F are firstly mapped to the vectors 000, 100, 110, 101 and later to the number 0,4,6,5 respectively).

Algorithm

Given the cardinality n of the ground-set, the corresponding basis for checking intervality is primarily formed. This is done with the help of the following procedure nex:

Suppose that a subset A of the ground-set E is already chosen. The procedure nex produces all possible combinations of a given length, which are composed of the elements of the set E-A. The elements of each particular combination are supposed to be added to the corresponding word. One zero is added at the end of the word. The same elements are added to the chosen set in the next step of iteration.

EXAMPLE. Let E={1,2,3,4}, A={2}. The word 20 is already produced. If the length 2 is given, then the following application of nex will produce the words 20130, 20140, 20340, in this order (the corresponding chosen sets will be {1,2,3}, {1,2,4}, {2,3,4} respectively).

The procedure nex is called at the four inner levels of a fivefold loop. The outer level corresponds to the combinations which produce all possible vectors (a,b,c), which satisfy the conditions quoted above. The required combination lengths at the four inner levels (from outside towards inside) are equal to a, b-a, c-b, 1 respectively.

* * *

Let G be a greedoid on an n-set (coded as given above) and suppose that the corresponding basis for testing greedoid intervality is already established. Each word of the basis determines some six (0-1) components of the vector v, which is associated to G. We ask whether these six components correspond to a non-interval critical 6-tuple (this happens iff there exist exactly one properly located zero and five units among the six components). If G is interval, then all g(n) words should be used. In a non-interval case, however, we replace the greedoid after we have found the first non--interval critical 6-tuple.

The above lemma is used to escape testing intervality of a full greedoid (i.e., a greedoid satisfying mc=n) in the case when the non-full greedoid in the same couple is non-interval.

Results

We give the table of the number of non-isomorphic interval greedoids with mc fixed on an n-set, for $0 \le mc \le n \le 5$:

n / mc	0	1	2	3	4	5
0	1	1	1	1	1	1
1		1	2	3	4	5
2			2	8	25	70
3				6	97	1072
4					34	2691
5						672
Σ	1	2	5	18	161	4511

REMARKS. It is obvious that all greedoids with $mc \le 2$ are interval. The number of non-isomorphic non-interval greedoids on 3,4,5 elements is equal to 2, 67, 20454 respectively (the last number suggests the conjecture that almost all greedoids are non-interval for large values of n).

The main diagonal of the table contains the number of non-isomorphic shelling structures (=alternative precedence structures = antimatroids ≒ full interval greedoids).

It seems that the probability of intervality decreases with the number of feasible 1-sets. Such a conclusion may be suggested by the following table of the number of non-isomorphic greedoids for n=5:

number of feasible 1-sets	1	2	3	4	5
interval greedoids, mc=3	68	1283	1189	147	4
all greedoids, mc=3	99	2992	6002	2486	329
interval greedoids, mc=4	34	366	242	28	1
all greedoids, mc=4	99	2992	6002	2486	329

* * *

In order to list some of the generated greedoids, we introduce the abbreviations "a",...,"j","A",...,"D","T", "1",...,"5" to denote:

a	b	c	d	e	f	g	h	i	j
12	13	14	15	23	24	25	34	35	45

A	B	C	D	T
123	124	134	234	ABCD

"n" replaces the family {{1 ,...,{n}} for $1\leq n\leq 5$.

The lists given below are sufficient for reconstruction of all the non-isomorphic interval greedoids on 3 and 4 elements. Given $n\in\{3,4\}$, we list together all non-isomorphic interval greedoids satisfying mc=n-1, which are erected form the same greedoid satisfying mc=n-2. The corresponding families of feasible (n-1)-sets are listed after the short lines. If such a family is followed by ":", then the corresponding greedoid satisfying mc=n is also interval, otherwise the family is followed by ",". Thus each listed family corresponds to two interval greedoids in the first case and to the only one in the second case.

EXAMPLE. The denotation "1abc-AB,ABC:" stands for three interval greedoids with the following families of feasible sets:

F1={∅,1,12,13,14,123,124}

(this interval greedoid may be also denoted by 1abc-AB,)

F2={∅,1,12,13,14,123,124,134}

F3={∅,1,12,13,14,123,124,134,1234}

(these two interval greedoids may be also denoted by 1abc-ABC:).

Given a greedoid on 5 elements satisfying mc=2, we give in brackets only the corresponding numbers of erected interval greedoids satisfying mc=3, mc=4 and mc=5 respectively. Thus the denotation "3abceg(41,106,15) "means that there are 41, respectively 106, respectively 15 non-isomorphic interval greedoids on 5 elements satisfying mc=3, respectively mc=4, respectively mc=5, which are all erected from the (interval) greedoid {0,1,2,3,12,13,14,23,25}.

LISTS

3 elements:

1-a:ab: 2-a:ab:be,abe: 3-ab,abe:

4 elements:

1a-A:AB: 1ab-A:AB:BC,ABC: 1abc-AB,ABC: 2a-A:AB:
2ab-A:AB:AC:BC,ABC: 2be-CD, 2abc-AB,AC,ABC:
2abe A:AB:AC:ABC:ACD:BCD,T: 2abf-AB:AD,ABC:ACD,T:
2abce-AB,AC,ABC:ABD,ACD,BCD,T: 2bcef-CD,
2abcef-AB,ABC,ACD,T: 3ab-BC, 3abc-BC,
3abe-A:AB:ABC:BCD,T: 3abce-AB,ABC:ABD,BCD,T:
3abfh-BC, 3abcef-AB,ABC,ACD,BCD,T: 3abcfh-BC,
3abcefh-ABC,BCD,T: 4abcef-AB, 4abcefh-ABC,T:

5 elements:

1a(3,8,6) 1ab(9,35,19) 1abc(9,22,8) 1abcd(4,3,1)
2a(3,8,6) 2ab(12,56,29) 2be(2,2,0) 2abc(15,42,13)
2abe(26,187,86) 2abf(29,181,66) 2abcd(6,5,1)
2abce(69,365,93) 2abcg(34,105,18) 2bcef(4,3,0)
2abcde(27,36,4) 2abcef(39,139,31) 2abceg(41,106,15)
2abcdef(38,40,4) 2bcdefg(2,1,0) 2abcdefg(11,7,0)
3ab(2,2,0) 3abc(5,4,0) 3abe(15,87,42) 3cfh(1,0,0)
3abcd(3,1,0) 3abce(45,228,66) 3abfh(5,4,0)
3abcde(19,26,4) 3abcef(55,225,45) 3abceg(41,106,15)
3abcfh(9,8,0) 3abcgi(8,4,0) 3abcdef(55,61,5)
3abcdfh(7,2,0) 3abcefh(41,171,37) 3abcefi(59,116,14)
3abfghi(3,1,0) 3cdfghi(1,0,0) 3abcdefg(19,11,1)
3abcdefh(48,66,7) 3abcdefi(42,32,3) 3abcfghi(7,2,0)
3abcdefgh(38,24,2) 3abcdfghi(5,1,0) 3abcdefghi(14,7,1)
4abc(1,0,0) 4abcd(1,0,0) 4abfh(1,0,0) 4dgij(0,0,0)
4abcef(7,6,0) 4abcdef(7,2,0) 4abcefh(20,69,19)
4abcgij(1,0,0) 4abdfhj(1,0,0) 4abcdefg(5,1,0)
4abcdefh(24,34,5) 4abcdgij(1,0,0) 4abcefij(5,2,0)
4abcdefgh(24,17,2) 4abcdefij(6,1,0) 4abdfghij(1,0,0)
4abcdefghi(17,9,1) 4abcdefgij(4,1,0) 4abcdefghij(7,5,1)
5abcd(0,0,0) 5abcgij(0,0,0) 5abcdefg(1,0,0)
5abcdefij(1,0,0) 5abcdefghi(3,1,0) 5abcdefghij(4,3,1)

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The results concerning the interval greedoids on at most 4 elements were primarily obtained by hand. Our PASCAL program was tested on them. We used the computer DELTA 341 (PDP 11/34) at Institute of Mathematics, Novi Sad. The complete computing time was about two hours.

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REZIME

INTERVALNI GRIDOIDI NA SKUPOVIMA OD NAJVIŠE 5 ELEMENATA

Opisan je algoritam za ispitivanje intervalnosti gridoida. Primenjen je odgovarajući kompjuterski program na listu svih neizomorfnih gridoida na skupovima od najviše 5 elemenata, koja je dobijena u [1].

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