Univ. u Novom Sadu,
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 18,1,93-99(1988)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

A CONSTRUCTION OF ALL NON-ISOMORPHIC GREEDOIDS ON AT MOST 5 ELEMENTS

Dragan M. Acketa
Institute of Mathematics, 21000 Novi Sad
Dr Ilije Djuričića 4, Yugoslavia

ABSTRACT

An algorithm for producing all the non-isomorphic greedoids on the ground-set of a given cardinality n is described. We have performed the corresponding computer search for $n \leq 5$.

Preliminaries

A *greedoid* ([1]) on a finite ground-set E is a pair (E,F), where $F \subseteq 2^E$ (family of *feasible* sets) satisfies

(i) $\emptyset \in F$

(ii) $\forall X \in F,\ \exists x \in X$ so that $X-x \in F$

(iii) $X,Y \in F,\ |X|=|Y|+1,\ \exists x \in X-Y$ so that $Y \cup x \in F$.

Two greedoids (E,F_1) and (E,F_2) are *isomorphic* if there is a permutation p of E satisfying $p(F_1)=F_2$.

Let "mc" denote the maximal cardinality of a feasible set. An n-set is a set of cardinality n. Sets are denoted without brackets and commas.

AMS Mathematics Subject Classification (1980): Primary 05B35; Secondary 68E05.
Key words and phrases: Construction of non-isomorphic greedoids.

LEMMA. There is a natural bijection between the classes of greedoids on an n-set E ($n \geq 1$) satisfying mc=n-1 and mc=n respectively.

PROOF. Greedoids with $mc \geq n-1$ can be coupled together so that two greedoids in the same couple differ solely in feasibility of E. It is obvious that removing E from $\mathcal{F}$ and addition of E to $\mathcal{F}$-preserve (ii) and (iii).□

Given a family $\mathcal{F} \subseteq 2^E$, let "Aut($\mathcal{F}$)" denote the automorphism group of $\mathcal{F}$ (consisting of all those permutations of E, which fix $\mathcal{F}$ as a whole).

Algorithm

Given $E=\{1,...,n\}$, it is obvious that the empty greedoid is the only one with mc=0 and that there are n non-isomorphic greedoids with mc=1. According to the lemma, it remains to construct the non-isomorphic greedoids with mc=k, for k between 2 and n-1.

Consider a greedoid (E,$\mathcal{F}$) satisfying mc=k-1. Let $\mathcal{F}=F_0 \cup F_1 \cup \ldots \cup \ldots \cup F_{k-1}$, where "$F_j$" denotes the family of feasible j-sets. We should look for all non-isomorphic "greedoidity preserving" possibilities to add F_k to $\mathcal{F}$.

The feasible sets are represented as combinations without repetitions (i.e., as arrays of increasing integers). Let $1 \leq b \leq \binom{n}{k}$.

Then all $\binom{\binom{n}{k}}{b}$ combinations of length b of combinations of length k of elements of E may be viewed as the complete list of candidates of length b for F_k.

EXAMPLE. Given n=5, b=3 and k=2, the complete list of 120 candidates begins with {12,13,14} and finishes with {34,35,45}.

Each candidate C for F_k should pass three successive tests before we keep C and include $(E, \mathcal{F} \cup C)$ into the list of all non-isomorphic greedoids with mc=k, which are obtained by erection from $(E, \mathcal{F})$:

if (F_{k-1}, C) satisfies (ii) then
  if (F_{k-1}, C) satisfies (iii) then
    if C is lexicographically then first w.r.t. Aut $(\mathcal{F})$ then
      keep C.

Note that the tests for the conditions (ii) and (iii) are "local"; they use only F_{k-1} from the whole family $\mathcal{F}$.

Let $F_{k-1}=\{A(1),\dots,A(a)\}$ and $C=\{B(1),\dots,B(b)\}$.

Testing (ii): For each j, $1 \le j \le b$, ask whether some of the sets a(1),...,A(a) is included in B(j). If the answer is negative for some j, then (ii) is not satisfied.

Testing (iii): For each i, $1 \le i \le a$, ask whether there exists some $x \in B(j)$, for all j, $1 \le j \le b$, such that $A(i) \cup x$ belongs to C. We construct the auxiliary set $X(i)=\{x \mid A(i) \cup x \in C\}$. It suffices to check whether $X(i) \cap B(j)$ is non-empty for all j, $1 \le j \le b$. If the answer is negative for some i, then (iii) is not satisfied.

The last test is intended to prevent arising mutually isomorphic greedoids. Given the family $\mathcal{F}$, the group Aut$(\mathcal{F})$ is generated only once and stored until all the candidates for F_k are considered.

Given a non-identical permutation p from Aut$(\mathcal{F})$, let "C_p" denote the family of k-sets, which is obtained by the lexicographical reordering of the family p(C), where "p(C)" denotes the image of C under p. If C_p lexicographically precedes C for some p, then C is not lexicographically the first w.r.t. Aut$(\mathcal{F})$.

Example

Let n=5, b=4, k=3 and $F=\{0,1,2,12,13,14,15\}$.

None of the candidates C for F_3 containing any of the sets 234,235,245,345 satisfies (ii). This reduces the number of candidates from

$$\binom{10}{4} = 210 \text{ to } \binom{6}{4} = 15.$$

Twelve of the remaining candidates for F_k do not satisfy (iii). Thus given C={124,124,125,134} and i=j=4, we have that $X(i) \cap B(j) = 5 \cap 134 = \emptyset$. Following the lexicographical order of the 15 candidates, the corresponding couples (i,j) are respectively: (4,4), (3,4), (2,4), (4,2), (4,1), none, (3,2), none, (3,1), (1,4), none, (2,2), (2,1), (1,3), (1,2).

The last three candidates for F_k are K={123,124, 135,145}. L={123,125,134,145} and M={124,125,134,135}. The group Aut(F) consists of those six permutations of E={1,...,5}, which have fixed points 1 and 2. Let p and q respectively denote the transpositions (45) and (35) from Aut(F). Then p(L)={123,124,135,154} and a(M)={124,123,154, 153}, while $L_p = M_q = K$. It follows that only K is lexicographically the first w.r.t. Aut(F). The greedoids (E,F∪L) and (E,F∪M) are isomorphic to (E,F∪K). Thus (E,F∪K) is the only one left in the list of non-isomorphic greedoids with mc=3, which have four feasible 3-sets and which are obtained by erection from (E,F).

Results

Let g(n) denote the number of non-isomorphic greedoids on n elements. We list the values of g(n) for $n \le 5$, as well as their partitioning w.r.t. mc:

mc \ n	0	1	2	3	4	5
0	1	1	1	1	1	1
1		1	2	3	4	5
2			2	8	25	70
3				8	99	1720
4					99	11908
5						11908
g(n)	1	2	5	20	228	25612

In order to list some of the generated greedoids, we introduce the abbreviations "a",...,"j", "A",...,"D", "T", "1",...,"5" to denote:

a	b	c	d	e	f	g	h	i	j
12	13	14	15	23	24	25	34	35	45

A	B	C	D	T
123	124	134	234	ABCD

"n" replaces the family {{1},...,{n}} for $1 \leq n \leq 5$.

The lists given below are sufficient for reconstruction of all the non-isomorphic greedoids on 3 and 4 elements. Given $n \in \{3,4\}$, we list together all non-isomorphic greedoids satisfying mc=n-1, which are erected from the same greedoid satisfying mc=n-2 (the corresponding families of feasible (n-1)-sets follow the short lines and are mutually separated by commas). Thus "1a-A,AB" is the abbreviation for the greedo- {0,1,12,123} and {0,1,12,123,124} (note that the corresponding two greedoids satisfying mc=4 (i.e.,including 1234), as well as the greedoid {0,1,12} may be also derived here).

Given a greedoid on 5 elements satisfying mc=2, we give in brackets only the corresponding numbers of erected greedoids satisfying mc=3 and mc=4 respectively. Thus the denotation "3abceg(41,321)" means that there are 41, respectively 321, non-isomorphic greedoids on 5 elements satisfying mc=3, respectively mc=4, which are all erected from the greedoid {0,1,2,3,12,13,14,23,25}.

LISTS

3 elements: 1-a,ab 2-a,ab,be,abe 3-ab,abe

4 elements:

1a-A,AB 1ab-A,AB,BC,ABC 1abc-AB,ABC 2a-A,AB
2ab-A,AB,AC,BC,ABC 2be-A,AC,CD,ACD 2abc-AB,AC,ABC
2abe-A,AB,AC,ABC,ACD,BCD,T 2abf-AB,AD,ABC,ACD,T
2abce-AB,AC,ABC,ABD,ACD,BCD,T 2bcef-AB,CD,ABC,ACD,T
2abcef-AB,ABC,ACD,T 3ab-A,AB,BC,ABC 3abc-AB,BC,ABC
3abe-A,AB,ABC,BCD,T 3cfh-BC,BCD 3abce-AB,ABC,ABD,BCD,T
3abfh-AD,BC,ABC,ABD,BCD,T 3abcef-AB,ABC,ACD,BCD,T
3abcfh-BC,ABC,ABD,BCD,T 3abcefh-ABC,BCD,T 4abc-AB,ABC
4abfh-AD,ABC,T 4abcef-AB,ABC,ACD,T 4abcefh-ABC,T

5 elements:

1a(3,8) 1ab(9,42) 1abc(9,38) 1abcd(4,11) 2a(3,8) 2ab(12,65)
2be(9,42) 2abc(15,77) 2abe(26,200) 2abf(29,246) 2abcd(6,24)
2abce(69,624) 2abcg(34,272) 2bcef(26,198) 2abcde(27,213)
2abcef(39,282) 2abceg(41,321) 2abcdef(38,283) 2bcdefg(14,77)
2abcdefg(11,60) 3ab(9,42) 3abc(15,77) 3abe(15,96) 3cfg(9,38)
3abcd(8,32) 3abce(45,367) 3abfh(38,320) 3abcde(19,127)
3abcef(55,453) 3abceg(41,321) 3abcfh(61,480) 3abcgi(44,331)
3abcdef(55,454) 3abcdfh(38,283) 3abcefh(41,278) 3abcefi(59,434)
3abfghi(24,157) 3cdfghi(14,77) 3abcdefg(19,124)
3abcdefh(48,350) 3abcdefi(42,290) 3abcfghi(51,352)
3abcdefgh(38,261) 3abcdfghi(32,190) 3abcdefghi(14,68) 4abc(9,38)
4abcd(6,24) 4abfh(16,107) 4dgij(4,11) 4abcef(39,282)
4abcdef(38,283) 4abcefh(20,107) 4abcgij(18,113) 4abdfhj(24,157)
4abcdefg(19,124) 4abcdefh(24,152) 4abcdgij(15,93)
4abcefij(33,205) 4abcdefgh(24,149) 4abcdefij(41,279)
4abdfghij(19,101) 4abcdefghi(17,97) 4abcdefgij(23,135)
4abcdefghij(7,29) 5abcd(4,11) 5abcgij(14,77) 5abcdefg(11,60)
5abcdefij(19,101) 5abcdefghi(14,68) 5abcdefghij(4,12)

* * *

The results on $n \leq 4$ were primarily obtained by hand. Our PASCAL program was tested on them. We used the computer DELTA 341 (PDP 11/34). The computing time for n=5 was about six hours.

REFERENCE

[1] Korte, B., Lovasz, L., *Greedoids - a structural framework for the greedy algorithm*, Report No.82230-Or, Inst. of Oper. Research, Univ. of Bonn, 1982.

REZIME

JEDNA KONSTRUKCIJA SVIH NEIZOMORFNIH GRIDOIDA NA SKUPOVIMA OD NAJVIŠE 5 ELEMENATA

Opisan je algoritam za generisanje svih neizomorfnih gridoida nad nosačem date kardinalnosti n. Izvršeno je odgovarajuće kompjutersko pretraživanje za $n \leq 5$.

Received by the editors December 12, 1988.