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ON APPROXIMATELY AND BOUNDED HOMOMORPHISMS ON UNIFORM COMMUTATIVE SEMIGROUPS

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ABSTRACT

In this paper it is introduced the notion of ε-homomorphism from a commutative semigroup S_1 into a commutative uniform semigroup S_2. The existence of homomorphism, which is ε-close to an ε-homomorphism, is proved. It is introduced the triangular bounded homomorphism and the relation to different type of boundedness is proved.

1. INTRODUCTION

The topology of an uniform commutative semigroup with some families of pseudometric or families of triangular functionals are characterized in [7] and [10].

Special triangular functional is the negative definite function with nonnegative values - [1].

Using this analytical approach to the uniform semi-

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groups some results on homomorphisms on uniform semigroups in [7] and [8] are obtained.

In this paper we shall use also this analitical approach to obtain further characterizations of special homomomorphisms and functions which are close to homomorphisms.

We introduce the notion of ε-homomorphism and in the analogy with the results in [3] and [5] for real functions, we obtain the corresponding generalizations for homomorphisms and ε-homomorphisms on commutative uniform semigroups.

We introduce the triangular bounded homomorphism and we examine its relations with different type of boundedness.

2. APPROXIMATELY HOMOMORPHISM

We suppose that in this section S_1 is a commutative semigroup and S_2 is a commutative uniform semigroup with a topology given by a family of pseudometrics $\{d_i\}_{i\in I}$ which satisfy the condition

$$(d_+)\quad d_i(x+y,x'+y') \le d_i(x,x') + d_i(y,y')$$

$$(x,y,x',y' \in S_1;\ i \in I)\quad - [7],\ [10].$$

Definition 2.1. *A function $g : S_1 \to S_2$ is called* <u>*ε-homomorphism*</u> *if it satisfies the inequality*

$$(2.1)\qquad d_i(g(x+y),g(x) + g(y)) \le \varepsilon, \qquad (x,y \in S_1;\ i \in I).$$

g is <u>*approximately homomorphism*</u> *if it is ε-homomorphism for some $\varepsilon > 0$.*

We shall now prove that function g, $g : S_1 \to S_2$, which is ε-close to a homomorphism is a 3ε-homomorphism.

Theorem 2.1. *Let h be a homomorphism from S_1 to S_2. If g is a function from S_1 to S_2 such that for some $\varepsilon > 0$*

$$d_i(g(x),h(x)) \le \varepsilon, \qquad (x \in S_1,\ i \in I),$$

then g is 3ε-homomorphism.

Proof. We have for $x,y \in S_1$.

$$d_i(g(x+y),g(x)+g(y)) \le d_i(g(x+y),h(x+y)) +$$

$$+ d_i(g(x)+g(y),h(x)+h(y)) \le$$

$$\le d_i(g(x+y),h(x+y)) + d_i(g(x),h(x)) +$$

$$+ d_i(g(y),h(y)) \le 3\varepsilon .$$ □

With some additional conditions on S_2 we can prove also the opposite statement.

Theorem 2.2. *Let additionaly the family of metrics* $\{d_i\}_{i\in I}$ *in* S_2 *satisfy the condition*

$$(d_\alpha) \quad d_i(2x,2y) = 2^\alpha d_i(x,y) \quad (x,y \in S_2;\ i \in I)$$

for some $\alpha > 0$, S_2 *is complete and* S_2 *be 2-divisable semigroup, i.e. for each* $y \in S_2$ *there exists* $x \in S_2$ *such that* $2x = y$. *Then for each* ε*-homomorphism* $g : S_1 \to S_2$ *there exists a homomorphism* $h : S_1 \to S_2$ *such that*

$$(2.2) \qquad d_i(g(x),h(x)) \le \frac{1}{2^\alpha - 1}\varepsilon \ (x \in S_1;\ i \in I).$$

Proof. We denote by γ_2 the 2-root function* - [6] on S_2. We introduce a sequence $\{h_n\}$ of functions $h_n : S_1 \to S_2$ in the following way

$$h_n(x) = \gamma_2^n(g(2^n x)), \quad (x \in S_1,\ n \in \mathbb{N}),$$

where γ_2^n is the n-th composition of γ_2.

We shall show that the sequence $\{h_n\}$ is convergent and that the function h given by

$$\lim_{n\to\infty} d_i(h_n(x),h(x)) = 0, \quad (x \in S_1;\ i \in I)$$

is the desired additive function.

First, taking in (2.1) $x = y$ we have

$$d_i(g(2x),2g(x)) \le \varepsilon, \quad (x \in S_1;\ i \in I).$$

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* i.e. $\gamma_2(x+y) = \gamma_2(x) + \gamma_2(y)$ and $\gamma_2(2x) = x$.

Hence

$$(2.3) \qquad d_i(\gamma_2(g(x)),g(x)) < \frac{\varepsilon}{2}\alpha, \quad (x \in S_1;\ i \in I).$$

By induction using (2.3) we have

$$d_i(\gamma_2^n(g(2^n x)),g(x)) \le (2^{-\alpha} + \ldots + 2^{-\alpha n})\varepsilon, \quad (n \in \mathbb{N}).$$

Hence

$$(2.4) \qquad d_i(\gamma_2^n(g(2^n x)),g(x)) \le \frac{2^{-\alpha}}{1-2^{-\alpha}}\,\varepsilon, \quad (x \in S_1,\ n \in \mathbb{N},\ i \in I).$$

Now, we shall show that $\{h_n\}$ is a Cauchy sequence. We have by (2.4)

$$d_i(h_{n+m}(x),h_n(x)) = 2^{-\alpha n} d_i(\gamma_2^m(g(2^m \cdot 2^n x)),g(2^n x)) \le$$

$$\le 2^{-\alpha n}\,\frac{1}{2^{\alpha}-1}\,\varepsilon, \quad (x \in S_1;\ i \in I;\ n,m \in \mathbb{N}).$$

Letting $n \to \infty$, we obtain that $\{h_n\}$ is a Cauchy sequence. Since S_2 is a complete uniform semigroup, there exists a function $h : S_1 \to S_2$ such that

$$(2.5) \qquad d_i(h_n(x),h(x)) \to 0, \quad (x \in S_1,\ i \in I) \quad \text{as } n \to \infty.$$

Multiplying (2.1) with $2^{-\alpha n}$ we obtain

$$(2.6) \qquad d_i(\gamma_2^n(g(2^n(x+y))),\gamma_2^n(g(2^n x)) + \gamma_2^n(g(2^n y)))) \le 2^{-\alpha n}\varepsilon,$$

$$(x,y \in S_1;\ i \in I;\ n \in \mathbb{N}).$$

We have for the function h from (2.5)

$$d_i(h(x+y),h(x) + h(y)) \le d_i(h(x+y),h_n(x+y)) +$$

$$+ d_i(h_n(x+y),h_n(x) + h_n(y)) + d_i(h_n(x),h(x)) +$$

$$+ d_i(h_n(y),h(y)), \quad (x,y \in S_1;\ i \in I).$$

Letting $n \to \infty$ we obtain by (2.5) and (2.6)

$$d_i(h(x+y),h(x) + h(y)) = 0, \quad (x,y \in S_1;\ i \in I).$$

Hence

$$h(x+y) = h(x) + h(y), \qquad (x,y \in S_1),$$

i.e., h is additive.

Taking $n \to \infty$ in (2.4) we obtain (2.2). □

Remark 2.1. *For each pseudometric d which satisfies* (d_+), *as in the previous theorem, the number* α *in the condition* (d_α) *satisfies* $0 < \alpha \leq 1$.

Example 2.1. α-quasi norm q, $\alpha > 0$, introduced in [4] on a commutative group defines by $q(x-y)$ an pseudometric which satisfies (d_+) and (d_α).

Example 2.2. In each locally p-convex space X there exists p-homogeneus F-pseudonorm $\|\ \|$, i.e. such that

$$\|tx\| = |t|^p \|x\|, \qquad (t \in \mathbb{R};\ x \in X),$$

determining a topology equivalent as the original one - Theorem 3.1.4. [9]. Obviously for $0 < p \leq 1$, the corresponding pseudometric satisfies (d_+) and (d_α). If X is locally bounded space, i.e. an F*-space which contains a bounded neighbourhood of 0, then, for a certain p, $0 < p \leq 1$, there is a p-homogenous F-norm $\|\ \|$ equivalent to the original one - Theorem 3.2.1, [9].

With an analogous proof as was in [5] we obtain an extension theorem for homomorphisms on a semigroup.

Theorem 3. *Let* S_1 *be a commutative semigroup, and let* S *be a subsemigroup of* S_1 *such that for every* $x \in S_1$ *we have* $x + x \in S$. *Let* S_2 *be a 2-divisable commutative semigroup and let* $h : S \to S_2$ *be a homomorphism. Then there exists a unique homomorphism* $H : S_1 \to S_2$ *such that* $H(x) = h(x)$, $(x \in S)$, *which is given by*

$$H(x) = \gamma_2(h(2x)), \quad (x \in S_1),$$

where γ_2 *is the corresponding root function in* S_2.

3. BOUNDED HOMOMORPHISM

Let S_1 and S_2 be commutative uniform semigroups endowed with the induced families of triangular functionals F_1 and F_2, respectively - [7], [8].

Definition 3.1. *A homomorphism* $h : S_1 \to S_2$ *is* *triangular bounded* *(in short bounded) if for each* $f_2 \in F_2$ *there exist* $f_1 \in F_1$ *and a number* $M_{f_2} > 0$ *such that*

$$(3.1) \qquad f_2(h(x)) \le M_{f_2} f_1(x), \ (x \in S_1).$$

Remark 3.1. *In [1] it was introduced the notion of bounded function* $h : S_1 \to \mathbf{C}$ *with respect to an absolute value* α. *It is obvious that if* f *is a triangular function, then* $e^{f(x)}$ *is an absolute value.*

By [8], a subset A *of* S_1 *is functionally bounded if for each sequence* $\{\alpha_n\}$ *of nonnegative real numbers such that* $\alpha_n \to 0$, $\alpha_n f(x_n) \to 0$ *holds for each sequence* $\{x_n\}$ *from* A *and each* $f \in F_1$.

Now we have

Proposition 3.1. A bounded homomorphism $h : S_1 \to S_2$ map each functionally bounded set A from S_1 on a functionally bounded set in S_2.

With respect to the root boundedness we have

Proposition 3.2. If S_1 is n-divisable then a bounded homomorphism $h : S_1 \to S_2$ map each root bounded set A (see [8]) on a root bounded set in S_2.

Proposition 3.3. Each bounded homomorphism $h : S_1 \to S_2$ is continuous.

Now it is easy to show that the oposite of Proposition 1 is not true. Namely, the identity map I from a normed vector space X with the weak topology to the same normed vector space with the norm topology is functionally bounded, since they have the same bounded sets but not continuous. So by Proposition 3 the map I is not triangular bounded.

We can formulate the Definition 1. also in the following equivalent from: a homomorphism $h : S_1 \to S_2$ is triangular bounded if there exists a map $\psi : F_1 \to F_2$ and a number $M > 0$ such that

$$\psi(f)(h(x)) \le Mf(x), \quad (x \in S_1). \tag{3.2}$$

Now, we denote with $L_\psi(S_1,S_2)$ the set of all homomorphisms $h : S_1 \to S_2$ which satisfy (3.2).

<u>Proposition 3.4.</u> The functional

$$F_{f,\psi}(h) := \sup_{x\in S_1} \frac{\psi(f)(h(x))}{f(x)}$$

(if S_1 is a semigroup with a neutral element, then in the preceding supremum we exlude this neutral element) is a triangular functional on the semigroup $L_\psi(S_1,S_2)$ endowed with the usual operation

$$(h_1 + h_2)(x) = h_1(x) + h_2(x), \quad (x \in S_1)$$

for $h_1,h_2 \in L_\psi(S_1,S_2)$.

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REZIME

O APROKSIMATIVNIM I OGRANIČENIM HOMOMORFIZMIMA NAD KOMUTATIVNIM UNIFORMNIM POLUGRUPAMA

U radu je uveden pojam ε-homomorfizma h sa komutativne polugrupe S_1 u komutativnu uniformnu polugrupu S_2. Uz neke dodatne pretpostavke na S_2 dokazana je egzistencija homomorfizma, koji je ε-blizu datom ε-homomorfizmu. Uveden je i pojam trougaono ograničenog homomorfizma i ispitana je njegova veza sa raznim tipovima ograničenosti.

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