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ADDITIVE SELECTIONS OF ADDITIVE SET-VALUED FUNCTIONS

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ABSTRACT

Assume that X and Y are vector spaces, K is a cone in X and $F:K\to 2^Y\setminus\{\emptyset\}$ is an additive set-valued function. We prove that if for some $x_0 \in \mathrm{ri}K$ the set $F(x_0)$ has an extremal point, then there exists an additive selection of F.

Theorems on the existence of selections having some "nice" topological or algebraic properties play an important role in the theory of set-valued functions and have numerous applications. In the case of additive set-valued functions natural and interesting is the question concerning the existence of additive selections. There are some results relating to this question. In [6] Rådström has proved that every additive set-valued function $F:(0,\infty)\to CC(Y)$ (here CC(Y) denotes the family of all non-empty, convex and compact subsets of Y), where Y is a locally convex Hausdorff space, has an additive selection. In the proof of this theorem he used the technique of support functions. Other constructions of additive selections

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one can find in a paper by Przesławski [5] (for additive set-valued functions taking values in $CC(R^n)$) and in an earlier paper by the author [4] (for additive set-valued functions $F: [0,\infty)\to CC(Y)$, where Y is a Hilbert space). The selection in [5] is defined by means of the so-called Steiner point whereas the selection in [4] is determined by the greatest elements of the values of F in the sense of a lexicographic order.

In the present paper we shall give another construction of additive selections. The method used makes it possible to omit the assumption that values of F are convex and compact; moreover, no topological structure of the space Y is required.

Let X and Y be arbitrary real vector spaces and assume that K is a cone in X (i.e. K is a subset of X satisfying the conditions $K+K\subset K$ and $tK\subset K$ for all $t\in(0,\infty)$). A set-valued function $F:K\to 2^Y\setminus\{\emptyset\}$ is said to be additive iff

$$F(x_1+x_2) = F(x_1)+F(x_2) \quad \text{for all} \quad x_1,\ x_2\in K.$$

Given a non-empty set $A\subset X$ we denote by lin A the affine space generated by A. We say that a point $a\in X$ belongs to the relative algebraic interior of A (and write $a\in\text{ri}A$) if for every $x\in\text{lin }A$ there exists an $\varepsilon>0$ such that $\{tx+(1-t)a:\ t\in(-\varepsilon,\varepsilon)\}\subset A$. A point $a\in A$ is said to be an extremal point of A if there are no two different points $a_1,a_2\in A$ and no number $t\in(0,1)$ such that $a = ta_1+(1-t)a_2$. The set of all extremal points of A is denoted by $\text{Ext}A$.

The following theorem holds:

Theorem. *Let* X *and* Y *be two real vector spaces and let* K *be a cone in* X. *Assume that* $F:K\to 2^Y\setminus\{\emptyset\}$ *is an additive set-valued function,* $x_o\in\text{ri}K$ *and* $p\in\text{Ext }F(x_o)$. *Then there exists exactly one additive selection* $f:K\to Y$ *of* F *such that* $f(x_o) = p$.

We shall start with the following lemma which is basic to the proof of our theorem.

Lemma 1. *Let* A *and* B *be subsets of a real vector space. If* $p \in \mathrm{Ext}(A+B)$, *then there exist exactly one point* $a \in A$ *and exactly one point* $b \in B$ *such that* $a+b=p$. *Moreover,* $a \in \mathrm{Ext}A$ *and* $b \in \mathrm{Ext}B$.

This result was originally given by Minkowski [3] for convex subsets of 3-dimensional space. For compact subsets of a locally convex Hausdorff space it is proved in [2] by Husain and Tweddle.

Proof. Let $p \in \mathrm{Ext}(A+B)$ and assume that $p=a_1+b_1=a_2+b_2$, where $a_1,a_2 \in A$ and $b_1,b_2 \in B$. Then

$$p = \frac{1}{2}(a_1+b_2) + \frac{1}{2}(a_2+b_1),$$

which implies that $a_1+b_2=a_2+b_1$. Hence, in view of the equality $a_1+b_1=a_2+b_2$, we obtain $a_1=a_2$ and $b_1=b_2$. Thus, there exist a unique $a \in A$ and a unique $b \in B$ such that $p=a+b$. We shall show that $a \in \mathrm{Ext}A$ and $b \in \mathrm{Ext}B$. For the indirect proof suppose that $a \notin \mathrm{Ext}A$. Then there exist points $a_1,a_2 \in A$, $a_1 \neq a_2$, and a number $t \in (0,1)$ such that $a = ta_1+(1-t)a_2$. Hence $p = t(a_1+b)+(1-t)(a_2+b)$, which contradicts the fact that $p \in \mathrm{Ext}(A+B)$. Therefore $a \in \mathrm{Ext}A$. An analogous reasoning shows that b ExtB. ::

Lemma 2. *Let* A *be a subset of a real vector space. If* $A+A=A$ *then* $\mathrm{Ext}A \subset \{0\}$.

Proof. Assume that $p \in \mathrm{Ext}A$. Since $A=A+A+A$, there exist points $a_1,a_2,a_3 \in A$ such that $p = a_1+a_2+a_3$. Obviously, we can write $p = a_1+(a_2+a_3)=(a_1+a_2)+a_3$, where $a_2+a_3 \in A$ and $a_1+a_2 \in A$. Hence, in view of Lemma 1, $a_1=a_1+a_2$, i.e. $a_2=0$. In an analogous way one can show that $a_1=0$ and $a_3=0$. Consequently $p=0$, which was to be proved. ::

Proof of Theorem. Let $U:=K \cap (x_o-K)$. If $u \in U$, then $x_o-u \in U$ and, by the additivity of F, $F(x_o) = F(u)+F(x_o-u)$. Then on account of Lemma 1, there exist a unique point $p_u \in F(u)$ and a unique point $p_{x_o-u} \in F(x_o-u)$ such that

$$(*) \qquad p = p_u + p_{x_o-u}.$$

Define a function $f_o:U\to Y$ putting $f_o(u):=p_u$, $u\in U$. Of course, $f_o(u)\in F(u)$ for all $u\in U$. We shall show that f_o is additive on U, that is $f_o(u+v) = f_o(u)+f_o(v)$ provided $u,v\in U$ and $u+v\in U$. Since $p\in F(x_o) = F(u)+F(v)+F(x_o-u-v)$, there exist points $a\in F(u)$, $b\in F(v)$ and $c\in F(x_o-u-v)$ such that $p=a+b+c$. By the uniqueness of the representation (*), we infer that $a=p_u$, because $a\in F(u)$ and $b+c\in F(v)+F(x_o-u-v) = F(x_o-u)$. Similarly $b=p_v$, because $b\in F(v)$ and $a+c\in F(x_o-v)$. Since $a+b\in F(u+v)$ and $c\in F(x_o-u-v)$, we have also $a+b=p_{u+v}$. Hence, using the definition of f_o, we obtain

$$f_o(u+v)=p_{u+v}=a+b=p_u+p_v=f_o(u)+f_o(v),$$

which was to be shown. Now, by a theorem of Dhombres and Ger [1], there exists an additive function $f:K\to Y$ such that $f|_U=f_o$. We shall show that f is a selection of F. To this aim fix an $x\in K$. Since $x_o\in \mathrm{ri}K$ and $x+x_o\in K$, there exists an $\varepsilon>0$ such that $t(x+x_o)+(1-t)x_o\in K$ for every $t\in(-\varepsilon,\varepsilon)$. Taking a natural number $n > \frac{1}{\varepsilon}$, we get

$$-\frac{1}{n}(x+x_o)+(1+\frac{1}{n})x_o\in K,$$

whence $\frac{x}{n}\in x_o-K$. Consequently, $\frac{x}{n}\in U$, because $\frac{x}{n}\in K$, too. Now we have

$$f(x) = nf(\tfrac{x}{n}) = nf_o(\tfrac{x}{n})\in nF(\tfrac{x}{n})\subset F(\tfrac{x}{n})+\ldots+F(\tfrac{x}{n}) = F(x)$$

which means that f is a selection of F.

Notice that this selection passes through the point p. Indeed, if $0\in K$, then $x_o\in U$ and $f(x_o)=p_{x_o}$, where $p_{x_o}+p_o=p$. Since $p_o\in \mathrm{Ext}\,F(0)$ and, in view of Lemma 2, $\mathrm{Ext}\,F(0)\subset\{0\}$, we have have $p_o=0$. Thus $p_{x_o}=p$. If $0\notin K$, then taking a point $u\in U$ and using (*), we obtain

$$f(x_o)=f(u)+f(x_o-u)=f_o(u)+f_o(x_o-u)=p_u+p_{x_o-u}=p;$$

Now, assume that $g:K\to Y$ is another additive selection of F such that $g(x_o)=p$. Then, for every $u\in U$ we have $g(u)\in F(u)$, $g(x_o-u)\in F(x_o-u)$ and $g(u)+g(x_o-u)=g(x_o)=p$. Hence $g(u)=p_u=f(u)$, because the representation (*) is unique. If x is an arbitrary element of K, we can find, as was noticed above, a natural number n such that $\frac{x}{n}\in U$. Then

$$g(x) = ng(\tfrac{x}{n}) = nf(\tfrac{x}{n}) = f(x),$$

which shows that the selection f is unique. This completes the proof.

Remarks 1. This theorem generalizes the theorem of Rådström mentioned at the beginning, because every compact set in a locally convex Hausdorff space has an extremal point.

2. The assumption that $x_o\in\text{ri}K$ is essential for the existence of the additive selection. For example the set-valued function $F:[0,\infty)\to 2^{\mathbf{R}}$ defined by

$$F(x) := \begin{cases} \{0\} , & x = 0 \\ (0,x)\cap Q, & x > 0, \end{cases}$$

where Q denotes the set of all rational numbers, is additive and $0\in\text{Ext}\,F(0)$. Nevertheless, F does not admit any additive selection.

3. If $p\in\text{Ext}\,F(x_o)$ or if $p\notin\text{Ext}\,F(x_o)$ but $x_o\notin\text{ri}K$, then there may exist more additive selections passing through p. Consider, for instance, the set-valued functions $F_1:\mathbf{R}\to 2^{\mathbf{R}}$ and $F_2:[0,\infty)\to 2^{\mathbf{R}}$ defined by

$$F_1(x) := R,\ x\in R \quad \text{and} \quad F_2(x) := \begin{cases} R\ , & x > 0 \\ \{0\}\ , & x = 0 \end{cases} .$$

Of course F_1 and F_2 are additive and functions of the form $f_c(x):=cx$, $x\in R$ (resp. $x\in[0,\infty)$), where c is a real constant, are their additive selections such that $f_c(0)=0$.

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REZIME

ADITIVNE SELEKCIJE ADITIVNIH SKUPOVNIH FUNKCIJA

Pretpostavimo da su X i Y vektorski prostori, K konus u X i $F:K\to 2^Y\setminus\{\emptyset\}$ aditivna skupovna funkcija. Dokazano je da ako za neko $x_0\in \text{ri}K$ skup $F(x_0)$ ima ekstremalnu tačku, tada postoji aditivna selekcija of F.

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