Univ. u Novom Sadu,
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 18,1,149-156 (1988)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

A COINCIDENCE THEOREM FOR MULTIFUNCTIONS

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ABSTRACT

In this paper a coincidence theorem for multifunctions is proved which generalizes the results obtained by O. Hadžić [1].

Let (X,d) be a metric space. We denote by $CB(X)$ the set of all the non-empty closed bounded subsets of (X,d) and by H the Hausdorff-Pompeiu metric on $CB(X)$

$$H(A,B) = \max\{\sup_{x\in A} d(x,B);\ \sup_{y\in B} d(y,A)\},$$

where $A,B\in CB(X)$ and

$$d(x,A) = \inf_{y\in A}\{d(x,y)\}.$$

Let $A,B\in CB(X)$ and $k>1$. In what follows, the following wellknown fact will be used: For each $a\in A$, there is a $b\in B$ such that

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AMS Mathematics Subject Classification (1980): 54H25.

Key words and phrases: Fixed point, multifunction.

$$d(a,b) \leq k\, H(A,B).$$

In two recent papers [1], [2], the following theorems have been proved.

Theorem 1. *Let* (X,d) *be a complete metric space,* S *and* T *continuous mappings from* X *into* X*,* A *a closed mapping from* X *into* $CB(SX\cap TX)$ *such that* $ATx = TAx$, $ASx = SAx$ *for every* $x\in X$ *and:*

(1) $H(Ax,Ay) \leq qd(Sx,Ty)$ *for every* $x,y\in X$ *where* $q\in(0,1)$.

Then there exists a sequence $\{x_n\}_{n\in N}$ *such that:*

1. *For every* $n\in N, Sx_{2n+1}\in Ax_{2n}; Tx_{2n}\in Ax_{2n-1}$.

2. *There exists* $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$.

3. $Tz\in Az;\ Sz\in Az$, [1].

Theorem 2. *Let* (X,d) *be a complete metric space,* S *and* T *continuous mappings from* X *into* X*,* A *and* B *closed mappings from* X *into* $CB(SX\cap TX)$ *such that* $ASx = SAx$, $BTx = TBx$ *for every* $x\in X$ *and*

(2) $$H(Ax,By) \leq q \max\{d(Sx,Ty);\ d(Sx,Ax);$$

$$d(Ty,By);\ \tfrac{1}{2}[(d(Sx,By)+d(Ty,Ax)]\}$$

for every $x,y\in X$ *where* $q\in(0,1)$. *Then, there exists a sequence* $\{x_n\}_{n\in N}$ *in* X *such that*

1. *For every* $n\in N, Sx_{2n+1}\in Bx_{2n};\ Tx_{2n}\in Ax_{2n-1}$.

2. *There exists* $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$.

3. Sz Az, Tz Bz, [2].

Now, we shall prove a similar coincidence theorem which generalizes the result of Hadžić and is independent of Theorem 2.

Remark 1. The following conditions are independent :

(a) There exists a constant $h, 0<h<1$, such that for each $x,y \in X$

$$H(Ax,By) \leq h \max\{d(Sx,Ty);\ d(Sx,Ax);\ d(Ty,By);\ \tfrac{1}{2}[d(Sx,By)+d(Ty,Ax)]\}.$$

(b) There exists a constant $h, 0<h<1$, such that for each $x,y \in X$

$$H^2(Ax,By) < h \max\{d^2(Sx,Ty);\ d(Sx,Ax).\ d(Ty,By);\ d(Sx,By).\ d(Ty,Ax);\ \tfrac{1}{2}[d(Sx,Ax).d(Sx,By) + d(Ty,Ax).d(Ty,By)]\}.$$

Example 1. Let $X = \{1,2,3,4\}$. Define a metric d for X by $d(1,4) = \frac{5}{2}$; $d(1,2) = d(1,3) = \frac{3}{2}$; $d(2,3) = \frac{4}{5}$; $d(3,4) = \frac{7}{4}$; $d(2,4) = 1$. Let A,B,S and T be the mappings on X such that

$$A(1) = 3;\ A(2) = A(3) = A(4) = 4$$

$$B(1) = B(2) = B(4) = 4;\ B(3) = 2$$

$$S = T = id.$$

Then A,B,S and T satisfy condition (a) for each $\frac{14}{15} \leq h < 1$ Ex.2; [3].

For x=1 and y=2

$$H^2(A(1),B(2)) = d^2(A(1),B(2)) =$$

$$= \frac{49}{16} > \max\{d(1,A(1)).d(2,B(2));\ d(1,B(2)).d(2,A(1));$$

$$\frac{1}{2}[d(1,A(1)).d(1,B(2))+d(2,A(1)).d(2,B(2)];\ d^2(1,2)\} =$$

$$= \frac{91}{40}.$$

Thus (b) is not satisfied.

Example 2. Let X = {1,2,3}. Define a metric d for X by

$$d(1,2) = 2;\ d(2,3) = 2;\ d(1,3) = \frac{3}{2}.$$

Let A,B,S and T be the mappings on X such that

$$A(1)=A(2)=A(3)=1;\ B(1)=B(3)=1;\ B(2)=3;\ S=T=id.$$

Then A,B,S, and T satisfy condition (b) for each $\frac{9}{16} \le h < 1$.

For x=1 and y=2 by (a)

$$H(A(1),B(2))=d(A(1),B(2))=d(1,3)=2\le h.\max\{d(1,A(1));$$

$$;\ \frac{1}{2}[d(1,B(2))+d(2,A(1))];\ d(1,2);\ d(2,B(2))\} = 2h$$

which implies $h \ge 1$. Thus (a) is not satisfied.

Theorem 3. *Let* (X,d) *be a complete metric space,* S *and* T *continuous mappings from* X *into* X,A *and* B *closed mappings from* X *into* CB(SX∩TX) *such that* ASx = SAx, BTx = TBx, *for every* x∈X *and*

(3) $H^2(Ax,By) \le h \max\{d(Sx,Ax).d(Ty,By);d(Sx,By).d(Ty,Ax);$

$$; \frac{1}{2}[d(Sx,Ax).d(Sx,By)+d(Ty,Ax).d(Ty,By)];\ d^2(Sx,Ty)\}$$

for every $x,y\in X$ *where* $h\in(0,1)$.

Then, there exists a sequence $\{x_n\}_{n\in N}$ *in* X *such that:*

1. *For every* $n\in N, Sx_{2n+1}\in Bx_{2n}, Tx_{2n}\in Ax_{2n-1}$.

2. *There exists* $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$.

3. $Sz\in Az$, $Tz\in Bz$.

Proof. Choose a real number k with

(4) $$1 < k^2 < \frac{1}{\sqrt{h}}.$$

Let $x_o\in X$. Since $Bx_o\subset SX$ there exists $x_1\in X$ such that $Sx_1\in Bx_o$. Then there exists an element $u_1\in Ax_1$ such that

$$d(u_1,Sx_1) \le k\ H(Ax_1,Bx_o)$$

and $u_2=Sx_3$ for some $x_3\in X$ because $Bx_2\subset SX$. Continuing in this feshion, we define a sequence $\{x_n\}_{n\in N}$ in X where $Tx_{2n}\in Ax_{2n-1}$ and $Sx_{2n+1}\in Bx_{2n}$ are such that

(5) $$d(Tx_{2n},Sx_{2n-1}) \le k\ H(Ax_{2n-1},Bx_{2n-2})$$

and

(6) $$d(Sx_{2n+1},Tx_{2n}) \le k\ H(Bx_{2n},Ax_{2n-1})$$

for every $n\in N$.

Now, we shall show that $\{Sx_{2n-1}\}_{n\in N}$ is Cauchy sequence. For this purpose, observe, first, that (5) yields

$$d^2(Tx_{2n},Sx_{2n-1}) \leq k^2H^2(Ax_{2n-1},Bx_{2n-2}) \leq$$

$$\leq k^2 \ h \max\{d(Sx_{2n-1},Ax_{2n-1}).d(Tx_{2n-2},Bx_{2n-2});$$

$$; d((Tx_{2n-2}.Ax_{2n-1}).d(Sx_{2n-1},Bx_{2n-2});$$

$$; \frac{1}{2}[d(Sx_{2n-1},Ax_{2n-1}).d(Sx_{2n-1},Bx_{2n-2}) +$$

$$+ d(Tx_{2n-2},Ax_{2n-1}).d(Tx_{2n-2},Bx_{2n-2})];$$

$$; d^2(Sx_{2n-1},Tx_{2n-2})\} \leq k^2 \ h \max \ d(Sx_{2n-1},Tx_{2n}).$$

$$.d(Tx_{2n-2},Sx_{2n-1}); \ d(Tx_{2n-2},Tx_{2n}).d(Sx_{2n-1},Sx_{2n-1});$$

$$; \frac{1}{2}[d(Sx_{2n-1},Tx_{2n}).d(Sx_{2n-1},Sx_{2n-1})+d(Tx_{2n-2},Tx_{2n}).$$

$$.d(Tx_{2n-2},Sx_{2n-1})]; \ d^2(Sx_{2n-1},Tx_{2n-2})\} =$$

$$= k^2 \ h \max\{d(Sx_{2n-1},Tx_{2n}).d(Tx_{2n-2},Sx_{2n-1});$$

$$; \frac{1}{2}.d(Tx_{2n-2},Sx_{2n-1}).[d(Tx_{2n-2},Sx_{2n-1})+d(Sx_{2n-1},Tx_{2n})];$$

$$; d^2(Sx_{2n-1},Tx_{2n-2})\} \leq k^2 \ h \ d(Tx_{2n-2},Sx_{2n-1}).$$

$$.\max[d(Sx_{2n-1},Tx_{2n});d(Tx_{2n-2},Sx_{2n-1})].$$

We have

$$d^2(Tx_{2n},Sx_{2n-1}) \leq k^2\sqrt{h}.d^2(Tx_{2n-2},Sx_{2n-1}),$$

and

$$d(Tx_{2n},Sx_{2n-1}) \leq k.\sqrt{h}.d(Tx_{2n-2},Sx_{2n-1})$$

$$\leq k^2.\sqrt{h}.d(Tx_{2n-2},Sx_{2n-1}),$$

or

$$d^2(Tx_{2n},Sx_{2n-1}) \leq k^2\ h\ d(Tx_{2n-2},Sx_{2n-1})\cdot$$

$$\cdot d(Tx_{2n},Sx_{2n-1}).$$

Then

$$d(Tx_{2n},Sx_{2n-1}) \leq k^2 h\ d(Tx_{2n-2},Sx_{2n-1}) \leq$$

$$\leq k^2 \sqrt{h}\ d(Tx_{2n-2},Sx_{2n-1}),$$

because $0<h<1$ and $k>1$ by (4).

Hence,

(7) $$d(Tx_{2n},Sx_{2n-1}) \leq k^2 \sqrt{h}\ d(Tx_{2n-2},Sx_{2n-1}).$$

In a similar manner, from (6), it follows that

(8) $$d(Sx_{2n+1},Tx_{2n}) \leq k^2 \sqrt{h}\ d(Tx_{2n},Sx_{2n-1})$$

By induction we obtain from (7) and (8)

$$d(Tx_{2n},Sx_{2n-1}) \leq (k^2 \sqrt{h})^n\ d(Sx_1,Tx_2)$$

and

$$d(Sx_{2n+1},Tx_{2n}) \leq (k^2 \sqrt{h})^n\ d(Tx_o,Sx_1)$$

for every $n\in N$. By (7) and (8), we have

$$d(Sx_{2n-1},Sx_{2n+1}) \leq d(Sx_{2n-1},Tx_{2n})+d(Tx_{2n},Sx_{2n+1}) \leq$$

$$\leq (k^2 \sqrt{h})^n[d(Sx_1,Tx_2)+d(Tx_o,Sx_1)].$$

Then, by a routine calculation, one can show that $\{Sx_{2n-1}\}_{n\in N}$ is a Cauchy sequence, and since (X,d) is complete, we have $\lim_{n\to\infty} Sx_{2n-1}=z$ for some $z\in X$. From (7), it follows that

$\lim_{n\to\infty} d(Tx_{2n}, Sx_{2n-1}) = 0$, hence $z = \lim_{n\to\infty} Tx_{2n}$. There remains to show that $Sz \in Az$ and $Tz \in Bz$. Indeed, the continuity of S implies that $Sz = \lim_{n\to\infty} STx_{2n}$. Further, since $Tx_{2n} \in Ax_{2n-1}$, it follows that $STx_{2n} \in SAx_{2n-1} = ASx_{2n-1}$, which, along with the fact that A is closed, implied that $Sz \in Az$. It can be similarly shown that $Tz \in Bz$.

Remark 2. In a similar manner, with Remark [2] it is proved that if S=T=id, the assumption of the closedness of A and B is superfluous.

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REZIME

O TAČKAMA KOINCIDENCIJE ZA VIŠEZNAČNA PRESLIKAVANJA

U ovom radu dokazana je teorema koincidencije za višeznačna preslikavanja koja uopštava rezultate O. Hadžić [1].

Received by the editors December 12, 1986.