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CYCLIC VECTOR VALUED GROUPOIDS

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Abstract. Cyclic (n,m)-groupoids which represent a generalization of cyclic n-ary quasigroups and semisymmetric quasigroups are defined and considered. If S is a nonempty set, m, n positive integers and F a mapping of S^n into S^m such that for all $x_1,\dots,x_{n+m} \in S$ $F(x_1,\dots,x_n) = (x_{n+1},\dots,x_{n+m})$ implies $F(x_2,\dots,x_{n+1}) = (x_{n+2},\dots,x_{n+m},x_1)$, then (S,F) is called a cyclic (n,m)-groupoid. Some properties of such (n,m)-groupoids are determined and it is proved that every cyclic (n,m)-groupoid can be generated by an n-ary groupoid satisfying an identity.

1. Introduction

Vector valued groupoids represent a convenient generalization of n-ary groupoids. Various classes of vector valued groupoids which generalize n-ary quasigroups, semigroups and some other structures were considered in [1], [2], [3], [7]. Here we shall consider a class of vector valued groupoids which represents a generalization of cyclic n-ary quasigroups and semisymmetric quasigroups and which is closely related to some combinatorial structures.

We shall use the following notation. The sequence $x_p, x_{p+1},\dots,x_q$ we denote by x_p^q . If p>q then x_p^q will be considered empty.

An n-ary groupoid (n-groupoid) (S,f) is called an n-quasigroup iff the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in S$ and every $i \in \{1,\dots,n\} = N_n$.

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Let (S,f) be an n-groupoid and $\sigma \in S_{n+1}$, where S_{n+1} is the symmetric group of degree n+1. If the n-operation f is uniquely solvable at the place $\sigma(n+1) = k$ (k-solvable), that is, for every $a_1^n \in S$ the equation

$$f(a_1^{k-1}, x, a_k^{n-1}) = a_n$$

has a unique solution, then by

$$f^\sigma(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = x_{\sigma(n+1)} \Leftrightarrow f(x_1^n) = x_{n+1}$$

an n-groupoid (S,f^σ) is defined. The operation f^σ is called a σ-parastrophe of f or simply parastrophe. f^σ is $\sigma^{-1}(n+1)$-solvable.

The set $\{(S,f_1),\ldots,(S,f_n)\}$ of n-groupoids is said to be orthogonal iff for every $(a_1^n) \in S^n$ there exists a unique $(b_1^n) \in S^n$ such that

$$f_i(b_1^n) = a_i, \ i = 1,\ldots,n.$$

Let S be a nonempty set, m, n positive integers and F a mapping of S^n into S^m. Then (S,F) is said to be an (n,m)-groupoid (or vector valued groupoid when it is not necessary to emphasize n and m). $|S|$ is called the order of (S,F). The n-ary operations $f_1,\ldots,f_m$ defined by

$$f_i(x_1^n) = y_i \Leftrightarrow (\exists y_1^{i-1}, y_{i+1}^m)\ F(x_1^n) = (y_1^m),\ i=1,\ldots,m,$$

are called the component operations (or components) of F.

Although every (n,m)-groupoid (S,F) can be interpreted as an algebra $(S,f_1,\ldots,f_m)$ with m n-ary operations, it is often more convenient to consider (n,m)-groupoids in the compact form as an algebra with one (n,m)-operation.

2. Cyclic (n,m)-groupoids

Definition 1. An (n,m)-groupoid (S,F) is called cyclic iff for every $x_1^{n+m} \in S$

$$F(x_1^n) = (x_{n+1}^{n+m}) \Rightarrow F(x_2^{n+1}) = (x_{n+2}^{n+m}, x_1)$$

Cyclic (n,m)-groupoids represent a generalization of cyclic n-groupoids and semisymmetric binary groupoids. For m = 1 a cyclic (n,m)-groupoid is a cyclic n-quasigroup (every cyclic n-groupoid is necessarily an n-quasigroup). Cyclic n-quasigroups were considered in [8] and their combinatorial applications in [9], [11]. For n = 2, m = 1 a cyclic (n,m)-groupoid becomes a well known semisymmetric binary quasigroup (a quasigroup satisfying the identity y(xy) = x is called semi-symmetric).

Cyclic (2,m)-groupoids are closely related to some combinatorial structures ([10]). A class of idempotent cyclic (2,m)-groupoids is equivalent to Mendelsohn designs and to decompositions of the complet directed graph K_v^* into arc disjoint elementary circuits of length m+2.

Definition 1 implies the following.

An (n,m)-groupoid (S,F) is cyclic iff for all $x_1^{n+m} \in S$ and every $i \in N_{n+m}$

$$F(x_1^n) = (x_{n+1}^{n+m}) \Leftrightarrow F(x_i^{n+i-1}) = (x_{n+i}^{n+m}, x_1^{i-1}),$$

where all indexes are taken modulo n+m.

Now we shall determine some properties of cyclic (n,m)-groupoids. We note first that if (S,F) is a cyclic (n,m)-groupoid, then for n=m F is a bijection and $F = F^{-1}$, that is, F^2 is the identity mapping (by F^{-1} we denote the inverse mapping of F). If S is a set and F the identity mapping of S^n, then (S,F) is a cyclic (n,n)-groupoid. Hence there exist cyclic (n,n)-groupoids of every order and every n.

Theorem 1. Let (S,F) be a cyclic (n,m)-groupoid and $f_1,\dots,f_m$ its components. Then

a) f_1 is 1-solvable and f_m is n-solvable,

b) f_m is a (1 2 ... n+1)-parastrophe of f_1.

P r o o f. a) Let $(a_1^n) \in S^n$. Then there exist unique $(y_1^m) \in S^m$ such that $F(a_1^n) = (y_1^m)$. Hence

$$F(y_m, a_1^{n-1}) = (a_n, y_1^{m-1}).$$

So, for every $(a_1^n) \in S^n$ the equation

$$f_1(y_m, a_1^{n-1}) = a_n, \tag{1}$$

has a solution y_m. If we assume that equation (1) has another solution z_m, then $f_1(z_m, a_1^{n-1}) = a_n$ implies that there exist z_1^{m-1} such that $F(z_m, a_1^{n-1}) = (a_n, z_1^{m-1})$ and by the cyclicity of F $F(a_1^n) = (z_1^m)$, hence $z_m = y_m$.

The proof is analogous for f_m.

b) Let $(x_1^n) \in S^n$. If $f_1(x_1^n) = y_1$, then $F(x_1^n) = (y_1^m)$ for some $(y_2^m) \in S^{m-1}$. Since $F(x_2^n, y_1) = (y_2^m, x_1)$, it follows $f_m(x_2^n, y_1) = x_1$, hence $f_1(x_1^n) = y_1$ implies $f_m(x_2^n, y_1) = x_1$. Analogously we get the inverse implication, f_1 is 1-solvable, which means that $f_1^{(12\dots n+1)} = f_m$.

Theorem 2. Let (S,F) be a cyclic (n,m)-groupoid, $n \le m$, with components $f_1,\dots,f_m$. Then $\{f_k,\dots,f_{k+n-1}\}$ is orthogonal for every $k \in N_{m-n+1}$.

P r o o f. For every $(a_1^n) \in S$, there exist unique $(y_1^m) \in S^m$ such that $F(a_1^n) = (y_1^m)$ which implies $F(y_{m-n+1}^m) = (a_1^n, y_1^{m-n})$. So, for every $(a_1^n) \in S^n$ the system

$$f_1(y_{m-n+1}^m) = a_1, \dots, f_n(y_{m-n+1}^m) = a_n,$$

has the unique solution $(y_{m-n+1}^m) \in S^n$, hence $\{f_1,\dots,f_n\}$ is an orthogonal system and analogously for any other n consequtive component operations $f_k,\dots,f_{k+n-1}$, $k = 1,\dots,m-n+1$.

Let (G,f) be the free n-groupoid on n generators $x_1,\dots,x_n$. We shall generate an infinite sequence of words in (G,f) in the following way:

$$w_1(x_1^n)=x_1,\ \dots\ ,\ w_n(x_1^n)=x_n,$$
$$w_{i+n}(x_1^n)=f(w_i(x_1^n),\ \dots\ ,w_{i+n-1}(x_1^n)),\quad i=1,2,\dots .$$

The identity of the form $w_k(x_1^n)=x_1$, $k>n$, is called k-cyclic identity and an n-groupoid satisfying this identity is called k-cyclic n-groupoid. k-cyclic binary quasigroups were considered in [4], [5], [6]. For $k=n+1$, $n\geq 2$, k-cyclic n-quasigroups are cyclic n-quasigroups from [8].

The definition of w_j implies the following identity;

$$w_j(w_{i+1-j}(x_1^n),\ \dots\ ,w_{i+n-j}(x_1^n)) = w_i(w_1(x_1^n),\ \dots\ ,w_n(x_1^n)),\quad 1\leq j\leq i\ . \tag{2}$$

Now we shall show that every (n,m)-groupoid can be defined by a single (n+m-1)-cyclic groupoid.

Let (S,F) be a cyclic (n,m)-groupoid and f the first component of (S,F). If $F(x_1^n)=(x_{n+1}^{n+m})$, then $x_{n+1}=w_{n+1}(x_1^n)$ and by the cyclicity of F

$$F(x_i^{n+i-1}) = (x_{n+i}^{n+m},x_1^{i-1}),\qquad i=1,\dots,n+m. \tag{3}$$

Hence for $i=2$ using (2) we get $x_{n+2}=w_{n+1}(x_2^{n+1})=w_{n+2}(x_1^n)$, and similarly for other values of i we obtain $x_{n+i}=w_{n+i}(x_1^n)$, $i=1,\dots,m$. Hence

$$F(x_1^n) = (w_{n+1}(x_1^n),\dots,w_{n+m}(x_1^n)). \tag{4}$$

Now from (3) for $i=2$ and (4) it follows

$$w_{n+m}(x_2^n,w_{n+1}(x_1^n)) = x_1$$

which by (2) gives $w_{n+m+1}(x_1^n) = x_1$.

Theorem 3. Let (S,f) be an n-groupoid. (S,f) is an n-groupoid satisfying the identity

$$w_{n+m+1}(x_1^n) = x_1.$$

iff the (n,m)-groupoid (S,F) defined by

$$F(x_1^n) = (w_{n+1}(x_1^n),\dots,w_{n+m}(x_1^n)) \tag{5}$$

is a cyclic (n,m)-groupoid .

P r o o f. Let (S,f) be an (n+m+1)-cyclic n-groupoid. If $x_1^n\in S$, then (5) is valid and using (2) we get

$$F(x_2^n,w_{n+1}(x_1^n)) = (w_{n+1}(x_2^n,w_{n+1}(x_1^n)),\dots,w_{n+m}(x_2^n,w_{n+1}(x_1^n))) =$$
$$(w_{n+2}(x_1^n),\dots,w_{n+m+1}(x_1^n)) = (w_{n+2}(x_1^n),\dots,w_{n+m}(x_1^n),x_1)$$

hence (S,F) is a cyclic (n,m)-groupoid.

The converse part of the theorem has been already proved.

Using cyclic (n,m)-groupoids by Theorem 3 some results on k-cyclic n-groupoids can be obtained and vice versa.

Theorem 4. Let (S,f) be an n-groupoid, k-1>n. Then for all $x_1^n \in S$

$$w_k(x_1^n) = x_1 \Leftrightarrow w_{n+1}(w_{k-1}(x_1^n),\ x_1^{n-1}) = x_n.$$

Proof. Let (S,f) satisfy $w_k(x_1^n) = x_1$. By the preceding theorem if F is defined by

$$F(x_1^n) = (w_{n+1}(x_1^n), \dots, w_{k-1}(x_1^n))$$

then (S,F) is a cyclic (n, k-n-1)-groupoid. Hence

$$F(w_{k-1}(x_1^n), x_1^{n-1}) = (x_n, w_{n+1}(x_1^n), \dots, w_{k-2}(x_1^n))$$

and consequently $w_{n+1}(w_{k-1}(x_1^n), x_1^{n-1}) = x_n$.

The inverse implication is proved analogously.

Similarly all previously obtained results and results from [10] on cyclic (2,m)-groupoids give the corresponding properties of n-groupoids satisfying the identity $w_k(x_1^n) = x_1$.

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REZIME

Definisani su i razmatrani ciklični (n,m)-grupoidi koji predstavljaju uopštenje cikličkih n-arnih kvazigrupa i polusimetričnih kvazigrupa. Ako je S neprazan skup, m i n prirodni brojevi a F preslikavanje S^n u S^m takvo da za svako $x_1,\dots,x_{n+m} \in S$ iz $F(x_1,\dots,x_n)=(x_{n+1},\dots,x_{n+m})$ sledi $F(x_2,\dots,x_{n+1})=(x_{n+2},\dots,x_{n+m},x_1)$, onda se (S,F) naziva ciklički (n,m)-grupoid. Određena su neka svojstva takvih (n,m)-grupoida i pokazano da se svaki ciklički (n,m)-grupoid može generisati n-arnim grupoidom koji zadovoljava jedan identitet.

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