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THE ASYMPTOTIC OF ELEMENTS BELONGING TO $\mathcal{D}_{L^p}$ AND $\mathcal{D}'_{L^p}$

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ABSTRACT

We know that a distribution $T\in\mathcal{D}'_{L^p}$, $1<p<\infty$ has the S-asymptotic related to $c(h)=1$ with the limit just zero; namely

$$\lim_{\|h\|\to\infty}\langle T(x+h),\phi(x)\rangle=0,\quad \phi\in\mathcal{D}.$$

We examine how fast $\langle T(x+h),\phi(x)\rangle$ can tend to zero when $\phi\in\mathcal{D}$ or $\phi\in\mathcal{D}_{L^q}$, $\frac{1}{p}+\frac{1}{q}\geq 1$.

INTRODUCTION

$\mathcal{D}_{L^p}$, $1\leq p<\infty$ is the set of smooth functions such that these functions and all their derivatives belong to

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L^p · $\dot{\mathcal{B}}$ is a subset of $\mathcal{D}_{L^\infty}$; a $\varphi\in\dot{\mathcal{B}}$ if and only if φ and all its derivatives tend to zero when $\|x\|\to\infty$. $\mathcal{D}_{L^p}\subset\mathcal{D}_{L^q}$ if $p\leq q$.

$\mathcal{D}'_{L^p}$, $1<p<\infty$ is the space of continuous linear functionals on $\mathcal{D}_{L^q}$, $1<q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. $\mathcal{D}'_{L^p}$ is the dual space of $\dot{\mathcal{B}}$.

In [1] we introduced and analysed the S-asymptotic in $\mathcal{D}'$. We shall use the following special case of it:

A distribution $T\in\mathcal{D}'$ has the S-asymptotic related to a $c(h)>0$, with the limit $U\in\mathcal{D}'$ if and only if there exists

$$(1)\qquad \lim_{\|h\|\to\infty} \langle T(x+h)/c(h),\phi(x)\rangle=\langle U,\phi\rangle\ ,\quad \phi\in\mathcal{D}.$$

The S-asymptotic can be also given in the space $\mathcal{D}'_{L^p}$, $1\leq p<\infty$ in an analogous way:

DEFINITION 1. *A distribution* $T\in\mathcal{D}'_{L^p}$ *has the* S-*asymptotic in* $\mathcal{D}'_{L^p}$, $p'<p$ *related to* $c(h)$ *with the limit* $U\in\mathcal{D}_{L^p}$, *if and only if there exists*

$$(2)\qquad \lim_{\|h\|\to\infty}\langle T(x+h)/c(h),\phi(x)\rangle=\langle U,\phi\rangle$$

for every $\phi\in\mathcal{D}_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$. *We write* $T(x+h)\overset{s}{\sim}c(h)\,U$ *in* $\mathcal{D}'_{L^p}$.

For the S-asymptotic in $\mathcal{D}'$ we know that U has the form $U = C\exp(\langle\alpha,x\rangle)$, $\alpha=(\alpha_1,\dots,\alpha_n)\in R^n$; $\langle\alpha,x\rangle=\sum_{i=1}^{n}\alpha_i x_i$

If we wish U to belong to $\mathcal{D}'_{L^p}$, $1\leq p<\infty$, then U can only be a constant C. Since $\mathcal{D}$ is dense in $\mathcal{D}_{L^q}$, $1\leq q<\infty$, then in Definition 1 U can only be a constant, as well.

In the following we shall use the well known relation ([2], T.II, p.22) for $T\in\mathcal{D}'$, $\phi\in\mathcal{D}$ or for $T\in\mathcal{D}'_{L^p}$, $\phi\in\mathcal{D}_{L^q}$, $\frac{1}{p}+\frac{1}{q}\geq 1$:

(3) $\langle T(x+h),\phi(x)\rangle=(T*\hat{\phi})(h)$, $\hat{\phi}(x)=\phi(-x)$.

We know that $(T*\phi)(h)$ belongs to $\mathcal{D}_{L^r}$, $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$.

In the case $\phi\in\mathcal{E}$ we have q=1 and r=p. Every element $\varphi\in\mathcal{D}_{L^r}$ has the property that $\varphi(x)\to 0$, $\|x\|\to\infty$ ([2], T.II,p.55) when $1\leq r<\infty$. We are interested how fast a φ can tend to zero. From relation (3) we can then conclude how fast $\langle T(x+h),\varphi(x)\rangle$ can tend to zero when $\|h\|\to\infty$, $T\in\mathcal{D}'_{L^p}$, $1\leq p$ or $T\in\mathcal{D}'$.

1. BEHAVIOUR OF ELEMENTS BELONGING TO $\mathcal{D}_{L^p}$, $1\leq p<\infty$ AT INFINITY

Taking care of the fact that $\varphi\equiv 0$ belongs to every $\mathcal{D}_{L^p}$, $1\leq p<\infty$, we do not have to analyze how the fastness, but rather the slowness depends on p. Also, starting from the fact that $\mathcal{D}_{L^p}\subset\mathcal{D}_{L^q}$, $p\leq q$, our questions are:

1. If there exist an element $\psi\in\mathcal{D}_{L^q}$ and a R>0 such that

$$\lim_{\|x\|\to\infty}\varphi(x)/\psi(x)=0,\ \|x\|\geq R,$$

does for every $\varphi\in\mathcal{D}_{L^p}$, p<q ?

2. For a fixed p is it possible to find a function c(x) such that $c(x)\to 0$, $\|x\|\to\infty$ and $|\varphi(x)/c(x)|\leq C_\varphi$, $\|x\|\geq R_\varphi$, for every $\varphi\in\mathcal{D}_{L^p}$? C_φ and R_φ are functions depending on φ.

To answer these questions, we shall construct a special function μ belonging to $\mathcal{D}_{L^1}$ and consequently to $\mathcal{D}_{L^p}$, $1\leq p<\infty$.

Let $\varepsilon>0$. By η_ε we denote the well known function ([3], p. 8) with the properties: $\eta_\varepsilon\in C^\infty$; $0\leq\eta_\varepsilon(x)\leq 1$; $\eta_\varepsilon(x)=1$, $x\in B(0,\varepsilon)$; $\eta_\varepsilon(x)=0$, $x\notin B(0,2\varepsilon)$; $|D^\alpha\eta_\varepsilon(x)|\leq K_\alpha\varepsilon^{-|\alpha|}$. B(0,ε) is the ball $\{x\in R^n, \|x\|<\varepsilon\}$, K_α is a constant depending on $\alpha, \alpha=(\alpha_1,\dots,\alpha_n), |\alpha|=\sum_{i=1}^{n}\alpha_i$.

For a function f such that $f(x)\to 0, \|x\|\to\infty$ and that for every $r>0$ there exists a $x_r, \|x_r\|\geq r$, $f(x_r)\neq 0$, we can find a sequence of positive numbers $\{r_n\}, \|r_n\|\to\infty$, such that $|f(x)|\leq 1/2^n, \|x\|>r_n$ and a sequence $\{y_n\}\subset R^n, \|y_n\|\geq r_n + 2\varepsilon$, $f(y_n)=a_n\neq 0$, $|a_n|\leq 1/2^n$. These two sequences can be made precise in such a way that the balls $B(y_n, 2\delta)$ are disjoint.

Now, the function μ we seek is

$$\mu(x)=\sum_{k=1}^{\infty}\lambda_k|a_k|\eta_\varepsilon(x-y_k),\quad 0<\lambda_k\leq k. \tag{4}$$

This series converges uniformly on R^n because of

$\lambda_k|a_k|\eta_\varepsilon(x-y_k)\leq k/2^k$, $x\in R^n$.

The following series, for every $\alpha\in R^n_+$

$$\sum_{k=1}^{\infty}\lambda_k|a_k|\,|D^\alpha\eta_\varepsilon(x-y_k)|$$

has the same property because of

$$\lambda_k|a_k|\,|D^\alpha\eta_\varepsilon(x-y_k)|\leq K_\alpha\varepsilon^{-|\alpha|}k/2^k,\quad x\in R^n.$$

Consequently $\mu\in C^\infty$.

But $\mu\in L^1$, as well.

Since $\mu(x)\geq 0$, it is enough to prove that

$\int_{B_n}\mu(x)dx$, $B_n=B(0,r_n)$,

is a Cauchy sequence of numbers.

$$\int_{B_{n+p}}-\int_{B_n}\mu(x)dx=\int_{B_{n+p}\setminus B_n}\sum_{k=1}^{n+p-1}\lambda_k|a_k|\eta_\varepsilon(x-y_k)\,dx$$

$$=\sum_{k=n}^{n+p-1}\int_{B(y_k,2\varepsilon)}\lambda_k|a_k|\eta_\varepsilon(x-y_k)\leq nQ/2^n$$

where Q is a constant. We have used the fact that for an x only one function $\eta_{\varepsilon}(x-y_k)$ can be different from zero.

In the same way, we can prove that $\mathcal{D}^{\alpha},\alpha\in R^n_+$ belongs to L^1, as well. This proves that $\mu\in\mathcal{D}_{L^1}$.

Now, we can return to our questions. The answer to both is negative. For the first one, we take in the construction of the function μ that $f=\psi$ and $\lambda_k=1$, $k\geq 1$. In this case the quotient $\mu(x)/\psi(x)$ can not tend to zero when $\|x\|\to\infty$, because of the fact that $\mu(y_n)/\psi(y_n)=1$, $n\geq 1$.

For the second question, we have to take $f=c$ and $\lambda_k=k$ in the construction of the function μ. Then, there is no function $c(x)$, $c(x)\to 0$, $\|x\|\to\infty$ such that for C_φ and R_φ

$$|\varphi(x)/c(x)|\leq C_\varphi, \|x\|\geq R_\varphi \text{ for every } \varphi\in\mathcal{D}_{L^p}, \infty>p\geq 1$$

because of the property of the fuction $\mu\in\mathcal{D}_{L^1}$ that $\mu(y_n)/c(y_n)=n$

2. BEHAVIOUR OF ELEMENTS BELONGING TO $\mathcal{D}'_{L^p}$, $p\geq 1$ AT INFINITY

Similar questions, as we had for a $\varphi\in\mathcal{D}_{L^p}$, we can ask for a $T\in\mathcal{D}'_{L^p}$, $1\leq p<\infty$, but for the S-asymptotic. So, we ask for a function $c(h)>0$, $c(h)\to 0$, $\|h\|\to\infty$ such that the set $\{T(x+h)/c(h), \|h\|\geq\beta\}$ whether $T\in\mathcal{D}'_{L^p}$ is weakly bounded in $\mathcal{D}'$ or in $\mathcal{D}'_{L^{p'}}$, $p\leq p'$? In a word, if there exists $c(h)>0$, $c(h)\to 0$, $\|h\|\to\infty$, such that

$$|\langle T(x+h)/c(h), \phi(x)\rangle|\leq M_\phi, \|h\|\geq\beta \text{ for every } T\in\mathcal{D}'_{L^p}$$

where β is a constant and M_ϕ depends on ϕ. Function ϕ belongs to $\mathcal{D}$ or to $\mathcal{D}_{L^p}$, $\frac{1}{p}+\frac{1}{q}$ 1, for a fixed $1\leq p<\infty$.

The answer to this question is negative, as well. To prove this we shall start from the fact that $\mathcal{D}_{L^1} \subset \mathcal{D}'_{L^p} \subset \mathcal{D}'$, $1<p$ and use once more our function μ given by relation (4) for the postive function $c(h)$. So, the constructed μ is a counterexample.

$$<\mu(x+h)/c(h),\phi(x)> = \int_{R_n} \frac{\mu(x+h)}{c(h)} \phi(x)\ dx$$

$$= \frac{1}{c(h)} \int_{R^n} \sum_{k=1}^{\infty} k\ a_k \eta_\varepsilon(x+h-y_k)\phi(x)\ dx$$

Since ϕ is any element belonging to $\mathcal{D}$ or to $\mathcal{D}_{L^p}$, we can choose $\phi(x)\geq 0$. Then, for $h=y_k$

$$<\mu(x+h)/c(h),\phi(x)> \geq k\ a_k/c(y_k) \int_{R^n} \eta_\varepsilon(x)\phi(x)\ dx$$

$$\geq k \int_{B(0,\varepsilon)} \phi(x)\ dx.$$

This shows that the sought function $c(h)$ does not exist.

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REZIME

ASIMPTOTIKA ELEMENTA IZ $\mathcal{D}_{L^p}$ i $\mathcal{D}'_{L^p}$

Poznato je da distribucija $T \in \mathcal{D}'_{L^p}$, $1 < p < \infty$ ima S-asimptotiku nula u odnosu na $c(h)=1$. Naime,

$$\lim_{\|h\| \to \infty} \langle T(x+h), \phi(x) \rangle = 0, \quad \phi \in \mathcal{D}.$$

U ovom radu izučavano je kojom brzinom može te-žiti nuli funkcija od h, $\langle T(x+h), \phi(x) \rangle$ kada $\phi \in \mathcal{D}$ ili $\phi \in \mathcal{D}_{L^p}$.

Polazeći od veze

$$\langle T(x+h), \phi(x) \rangle = (T * \check{\phi})(h), \quad \check{\phi}(x) = \phi(-x),$$

i činjenice da $(T*\phi)(h) \in \mathcal{D}_{L^r}$, gde je $r=p$ za $\phi \in \mathcal{D}$ i

$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ za $\phi \in \mathcal{D}_{L^q}$, $\frac{1}{p} + \frac{1}{q} > 1$, problem se svodi na ponašanje u beskonačnosti funkcija $\varphi(x) \in \mathcal{D}_{L^r}$, $r > 1$.

Pokazano je:

1. Ne postoji $\psi \in \mathcal{D}_{L^q}$ tako da je $\lim_{\|x\| \to \infty} \varphi(x)/\psi(x) = 0$ za svako $\varphi \in \mathcal{D}_{L^p}$, $p < q$.

2. Za utvrdjeno p ne postoji takva funkcija c(x), $c(x) \to 0$ $\|x\| \to \infty$ da je $|\varphi(x)/c(x)| < M_\varphi, \|x\| > R_\varphi; \phi \in \mathcal{D}_{L^p}$.

3. Ne postoji takva funkcija c(h), $c(h) \to 0$, $\|h\| \to \infty$ da je skup $\{T(x+h)/c(h), \|h\| > \beta\}$, $T \in \mathcal{D}'_{L^p}$ za utvrdjeno $p > 1$ slabo ograničen u $\mathcal{D}'$ ili $\mathcal{D}'_{L^{p'}}$, $p < p'$.

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