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A NON-LINEAR ORDER OF THE MARKOV PROCESS

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Abstract

The non-linear order $M < \infty$ of the Markov process $\{X(t),\ 0 \le t\}$ is defined as $M = \sup_s M(s)$ where $M(s)$ is the minimal number of random variables generating $\sigma\{E_s X(t),\ s \le t\}$ ($E_s(\cdot)$ is the conditional expectation with respect to $\sigma\{X(u),\ u \le s\}$). The relation between the non-linear and linear order of the Markov process is discussed. The latter is a modification of the Hida multiple Markov process.

Introduction

According to [1] a Gaussian process $\{X(t),\ 0 \le t\}$ is an N-ple Markov process if for any $0 < s \le t_1 < \ldots < t_N$, $E_s(X(t_i))$, $i=1,\ldots,N$ are linearly independent and for any $0 < s \le t_1 < \ldots < t_{N+1}$, $E_s(X(t_i))$, $i=1,\ldots,N+1$ are linearly dependent. $E_s(\cdot)$ is the conditional expectation with respect to the σ-field $\mathcal{F}_s(x)$ generated by $\{X(u), u \le s\}$. A small modification of this definition is given in [8]: Consider the mean-square linear closure. $\bar{\mathcal{H}}_b^a$ of $\{E_a(X(t)),\ a \le b \le t\}$. The Gaussian process $\{X(t),\ 0 \le t\}$ is an N-ple Markov process if for any $0 < a \le b$ the dimension of the space $\bar{\mathcal{H}}_b^a$ is exactly N.

The following modification of this definition seems to be convenient: Let $\bar{\mathcal{H}}_s$ be the mean-square linear closure of $\{E_s(X(t)),\ s \le t\}$ and let $N(s)$ be the dimension of $\bar{\mathcal{H}}_s$. The Gaussian process $\{X(t),\ 0 \le t\}$ is an N-ple Markov process if $N = \sup_s N(s)$ is finite.

For example, consider the process $\{\xi(t),\ 0 \le t \le 1\}$ defined by

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$$\xi(t) = W_1(t) + \varphi(t)W_2(t)$$

where $\varphi(t)$ is a version of the Cantor distribution function and $\{W_i(t),\ 0 \le t \le 1\}$, $i=1,2$ are independent Wiener processes. It is shown in [7] that $M(s)=2$ for s left-end points of removed intervals in the construction of the Cantor ternary set and $M(s)=3$ for the other s in $[0,1]$.

Let $\{X(t),\ 0 \le t\}$ be a second-order process with a separable space $\mathcal{H}_t(x)$ - the mean-square linear closure of $\{X(u),\ u \le t\}$ and let $\bigcap_t \mathcal{H}_t(x) = 0$. Let $P_s(\cdot)$ be the projection operator onto $\mathcal{H}_t(x)$. Let $N(s)$ be the dimension of the space $\tilde{\mathcal{H}}_s$ - the mean-square closure of $\{P_s(X(t)),\ s \le t\}$ and let $N = \sup_s N(s)$.

Definition 1. *The process $\{X(t),\ 0 \le t\}$ is the Markovian of the linear order N, if N is finite.*

Now, let $\{X(t),\ 0 \le t\}$ be the process with $E(X(t)) = 0$. Let $\mathfrak{F}_s$ be the σ-field generated by $\{E_s(X(t)),\ s \le t\}$. Denote by $M(s)$ the minimal number of random variables generating $\mathfrak{F}_s$ and let $M = \sup_s M(s)$.

Definition 2. *The process $\{X(t),\ 0 \le t\}$ is the Markovian of the non-linear order M, if M is finite.*

We find in [2] and [5] some motivations for Definition 2. In these papers the Gaussian Markov N-ple process $\{\xi(t),\ 0 \le t\}$ was considered for which, $0 \le s \le t$,

$$E_s(\xi(t)) = \sum_{j=0}^{N-1} a_j(s,t)\xi^{(j)}(s)$$

and the process $\{X(t), 0 \le t\}$ defined by $X(t) = \xi^n(t), n \ge 2$. It was shown that for $n=2$ ([2]) $E_s(X(t))$ is the rational function of $X(s), X'(s), \dots, X^{(N-1)}(s)$, and for $n \ge 3$ ([5])

$$E_s(X(t)) = X(s)R_0 + \left[X^{\frac{1}{n}}(s)\right]^{n-2} R_1 + \dots + \left[X^{\frac{1}{n}}(s)\right]^{n-4} R_2 + \dots ,$$

where $R_0, R_1, R_2, \dots$ is the rational function of $X(s)$, $X'(s), \dots, X^{(N-1)}(s)$.

We shall discuss the relation between the linear and non-linear order by some examples subsequently in the paper. These examples are mainly based on the Hermite polynomials $H_p(\xi_1, \xi_2, \dots, \xi_p)$ of the Gaussian variables $\xi_1, \xi_2, \dots$. Especially, the property that the conditional expectation and arbitrary Hermite polynomial commute:

$$E_s(H_p(\xi(t_1)),\dots,H_p(\xi(t_p))) = H_p(E_s(\xi(t_1)),\dots,E_s(\xi(t_p))), \qquad ([4]).$$

Some properties of the Markov border and the multiplicity of the Hermite polynomials of the Gaussian Markov process and fields are treated in [6].

First we shall remark that Definitons 1 and 2 are mutually free from one another. Indeed, for a Markov process $\{X(t)\}$ of the non-linear order M not to be of some linear order N it is sufficient that $E(X^2(t))=\infty$. Conversly, there are Markov processes of a linear order N, which are not Markov processes of some non-linear order M, as the following example shows.

Example 1. Let $\xi_1,\xi_2,\dots$ be a sequence of Gaussian variables with a complete stochastic dependence and let $\{X(t),\ 0 \le t\}$ be defined by

$$X(t) = \begin{cases} 0 \quad , & t < 1 \\ \sum_{p=1}^{[t]} H_p(\xi_p), & 1 \le t \end{cases} \quad , \quad (H_p(\xi) = \underbrace{H_p(\xi,\xi,\dots\xi)}_{p \text{ times}}) .$$

$\{X(t)\}$ is a process with orthogonal increments, so it is the Markovian of the linear order $N=1$. But, for $1 \le s < t$

$$E_s(X(t)) = E\left[\sum_{p=1}^{[t]} H_p(\xi_p)\,|\,\xi_1,\dots,\xi_{[s]}\right] = \sum_{p=1}^{[s]} H_p(\xi_p) +$$

$$+ E\left[\sum_{p=[s]+1}^{[t]} H_p(\xi_p)\,|\,\xi_1,\dots,\xi_{[s]})\right] = X(s) + \sum_{p=[s]+1}^{[t]} H_p(E(\xi_p\,|\,\xi_1,\dots,\xi_{[s]})).$$

Hence, $M(s) = [s]$ and $M = \infty$. □

It is a well-known fact that $E_s=P_s$ for Gaussian processes. The next two examples show that the class of processes for which $E_s= P_s$ is large.

Example 2. Let $\{\xi(t),\ 0 \le t\}$ be a Gaussian Markov process of orders $M=N=1$ and let $\{X(t),\ 0 \le t\}$ be defined by

$$X(t) = H_p(\xi(t)) .$$

Then $E_s(X(t)) = E(H_p(\xi(t)\,|\ \xi(u),\ u \le s)) = H_p(E_u(\xi(t))) = H_p(a(s,t)\xi(s)) =$
$= a^p H_p(\xi(s)) = a^p X(s)$.

Hence, $\{X(t)\}$ is the Markovian of non-linear order $M=1$. But $X(t) - a^p X(s)$ is orthogonal to $H_s(X)$: for all $u \le s$, $E((X(t) - a^p X(s))X(u)) =$
$EE_s(X(t)X(u)) - a^p E(X(s)X(u)) = a^p E(X(s)X(u)) - a^p E(X(s)X(u)) = 0$.
So, $P_s(X(t)) = a^p X(s) = E_s(X(t))$ and $N=1$. □

In the next example $E_s = P_s$, but $M \ne N$.

Example 3. We use some results from [3] and [8] in the following way:
Let the processes $\{Z_j(t),\ 0 \le t\}$, $j=1,\dots,n$ be mutually orthogonal widesense martingals and let the function $\psi(t)$ satisfy some continuity conditions. Consider the second-order continuous process $\{\xi(t),\ 0 \le t\}$ defined by

$$\xi(t) = \sum_{j=1}^{n} \psi^{j-1}(t) Z_j(t) .$$

We have that $H_t(\xi) = \bigoplus\sum_{j=1}^{n} H_t(Z_j)$, $0 \le t$, and $P_s(\xi(t)) = \sum_{j=1}^{n} \psi^{j-1}(t) Z_j(s)$.

It follows that $\{\xi(t)\}$ is the Markovian of the linear order $N=N(s)=n$.
If $\{W(t),\ 0 \le t\}$ is a Wiener process, then $\{H_k(W(t)),\ 0 \le t\}$, $k=1,\dots,n$ are mutually orthogonal wide-sense martingals. For the process $\{X(t),\ 0 \le t\}$ defined by

$$X(t) = \sum_{j=1}^{n} \psi^{j-1}(t) H_j(W(t)),$$

we have

$$P_s(X(t)) = E_s(X(t)) = \sum_{j=1}^{n} \psi^{j-1}(t) H_j(W(s)) .$$

So, the proces $\{X(t)\}$ is the Markovian of the linear order $N=n$ ($\mathcal{H}_s$ is the linear closure of $H_k(W(s))$, $k=1,\dots,n$) and of the non-linear order $M=1$ ($\mathfrak{F}_s$ is generated by the random variables $W(s)$). □

In the next example $M=N$, but $E_s \ne P_s$.

Example 4. Let $\{X(t),\ 0 \le t\}$ be defined by

$$X(t) = \sum_{k=1}^{m} t^{\alpha-k} H_k(W(t)) ,$$

where $\alpha > m$. As

$$E_s(X(t)) = E(X(t) \mid W(u),\ u \le s) = \sum_{k=1}^{m} t^{\alpha-k} H_k(E_s W(t)) = \sum_{k=1}^{m} t^{\alpha-k} H_k(W(s)) ,$$

the process $\{X(t)\}$ is the Markovian of the non-linear order $M=1$. The correlation function of $\{X(t)\}$ is, $s \le t$,

$$r(s,t) = E\, X(s)X(t) = E\Big[X(s)E_s X(t)\Big] = E\Big[\Big(\sum_{K=1}^{m} s^{\alpha-K} H_K(W(s))\Big)\Big(\sum_{j=1}^{m} t^{\alpha-j} H_j(W(s))\Big)\Big] =$$

$$= \sum_{K=1}^{m} s^{\alpha-K} t^{\alpha-K} EH_K^2(W(s)).$$

Since $EH_K^2(W(s)) = c_K s^K$, c_k are constants, we have

$$r(s,t) = s^{\alpha} \sum_{K=1}^{m} c_K t^{\alpha-K} = f(s)h(t)\ .$$

We conclude that $\{X(t)\}$ is the Markovian of the linear order $N=N(s)=1$. It is easy to see that

$$P_s(X(t)) = \frac{h(t)}{h(s)} X(s) \neq E_s(X(t)).\quad \square$$

Using the last two examples, it is clear that the relation between M and N may be arbitrary.

Example 5. Let $\{W_j(t),\ 0 \le t\}$, $j=1,\dots m$, $m \le n$ independent Wiener processes. The process $\{X(t),\ 0 \le t\}$ defined by

$$X(t) = \sum_{K=1}^{n} t^{\alpha-K} H_K(W_{j_K}(t)),\ j_K \in \{1,\dots,m\}\ ,$$

is the Markovian of $M=m$ and $N=1$.

The process $\{Y(t),\ 0 \le t\}$ defined by

$$Y(t) = \sum_{K=1}^{n} \psi^{K-1}(t) H_K(W_{j_K}(t))$$

is the Markovian of $M=m$ and $N=n$. $\square$

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Rezime

NELINEARNI RED MARKOVSKOG PROCESA

Definišemo nelinearni red $M < \infty$ Markovskog procesa $\{X(t),\ 0 \le t\}$ kao $M = \sup_s M(s)$ gde je $M(s)$ minimalni broj slučajnih promenljivih koje generišu $\sigma\{E_s X(t),\ s \le t\}$ (E_s je uslovno očekivanje u odnosu na $\sigma\{X(u),\ u \le s\}$. Diskutujemo odnos izmedu nelinearnog i linearnog reda Markovskog procesa. Ovaj poslednji je jedna modifikacija Hidinog višestrukog Markovskog procesa.

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