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THE QUASIASYMPTOTIC EXPANSION OF TEMPERED DISTRIBUTIONS AND THE STIELTJES TRANSFORM

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Abstract

The notion of the quasiasymptotic expansion of tempered distributions on a real line is given and several properties of this notion are presented. Also, an application on the Stieltjes transform of tempered distributions is given.

1. Introduction

In this paper we shall study the notion of the quasiasymptotic expansion of tempered distributions. This notion for tempered distributions supported by $[0,\infty)$ was introduced by Drožžinov and Zavi'alov see [1,III,10]. Pilipović also studied this notion in [5].

Denote by $\mathcal{S}$ the space of rapidly decreasing smooth functions defined on the real line R, supplied with the usual topology. Its dual, the space of tempered distributions is $\mathcal{S}'$ and $\mathcal{S}'_+$ is its subspace with elements supported by $[0,\infty)$.

Recall ([6]), a continuous positive function $L(x)$, $x \in (a,\infty)$, $a > 0$, is called slowly varying in infinity if for $\lambda > 0$

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1 .$$

For the properties of such functions we refer to [6].

The quasiasymptotic at ∞ of an $f \in \mathcal{S}'_+$ was studied by Vladimirov, Drožžinov and Zavi'alov [1]. Using this notion they obtained remarkable

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results. Pilipovic [3] extended this notion to the space $\mathcal{D}'$ of Schwartz distributions on the real line:

It is said that an $f \in \mathcal{D}'$ has the quasiasymptotic at $\pm\infty$ with respect to some positive continuous function $c(k)$, $k \in (a,+\infty)$, $a > 0$, if for some $g \in \mathcal{D}'$, $g \neq 0$,

$$\lim_{k\to\infty} \left\langle \frac{f(kx)}{c(k)}, \varphi(x) \right\rangle = \langle g(x), \varphi(x) \rangle, \quad \varphi \in \mathcal{D}. \tag{1-1}$$

In this case we write $f \overset{q}{\sim} g$ at $\pm\infty$ with respect to $c(k)$.

Theorem 1 ([3]). *Let $f \in \mathcal{D}'$ have the quasiasymptotic at $\pm\infty$ with respect to some positive continuous fuction $c(k)$, $k > a$. Then,*

(i) $f \in \mathcal{S}'$;

(ii) There are $\nu \in \mathbb{R}$ and a slowly varying function in infinity $L(k)$, $k > a$ such that $c(k) = k^{\nu} L(k)$, $k > a$. Moreover, g is a homogeneous distribution with the order of homogenity ν;

(iii) If $\nu \in \mathbb{R}\setminus(-\mathbb{N})$, then (1-1) holds in the sense of convergence in $\mathcal{S}'$ (for $\varphi \in \mathcal{S}$).

Let us recall that the family of homogeneuos distributions $f_{\nu+1}$, $\nu \in \mathbb{R}$, is defined by

$$f_{\nu+1}(x) = \begin{cases} \dfrac{H(x)x^{\nu}}{\Gamma(\nu+1)}, & \nu > -1 \\ f^{(n)}_{\nu+n+1}(x), & \nu \le -1,\ n+\nu > -1,\ n \in \mathbb{N}, \end{cases} \qquad (x \in \mathbb{R})$$

where H is the Heaviside function.

We also use the notion $H(x)x^{\nu} = x^{\nu}_{+}$, $H(-x)|x|^{\nu} = x^{\nu}_{-}$, $\nu > -1$.

The following theorem is necessary for our investigations.

Theorem A ([4]).

(i) Let F be a locally integrable function and $\nu \in \mathbb{R}$, $\nu > -1$, such that

$$\lim_{\substack{x\to+\infty \\ x\to-\infty}} \frac{F(x)}{|x|^{\nu} L(|x|)} = C_{\pm} \quad \text{where } (C_{+},C_{-}) \neq (0,0).$$

Then $F \overset{q}{\sim} g$ at $\pm\infty$ with respect to $k^{\nu}L(k)$, where $g(x) = \bar{C}_{+} f_{\nu+1}(x) + \bar{C}_{-} f_{\nu+1}(-x)$, $x \in \mathbb{R}$, $(\bar{C}_{+},\bar{C}_{-}) \neq (0,0)$.

(ii) If $f \overset{q}{\sim} g$ at $\pm\infty$ with respect to $k^{\nu}L(k)$, then $f^{(m)} \overset{q}{\sim} g^{(m)}$ at $\pm\infty$ with respect to $k^{\nu-m}L(k)$.

(iii) Let $f \in \mathcal{D}'$ and $f \overset{q}{\sim} g$ at $\pm\infty$ with respect to $k^{\nu}L(k)$, where $\nu \in \mathbb{R}\setminus(-\mathbb{N})$. There are $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and a continuous function F such that

$$m + \nu > -1,\ f = F^{(m)} \text{ and } \lim_{\substack{x\to+\infty \\ x\to-\infty}} \frac{F(x)}{|x|^{\nu}L(|x|)} = C_{\pm} \quad \text{where } (C_{+},C_{-}) \neq (0,0).$$

2. The quasiasymptotic expansion of an $f \in \mathcal{S}'$

We shall introduce the following family of distributions ([5]). Let $\nu \in \mathbb{R}$ and L be a slowly varying function defined on (a,∞), $a \geq 0$. We put

$$f_{L,a,\nu+1}(x) = \begin{cases} \dfrac{H(x-a)x^{\nu}L(x)}{\Gamma(\nu+1)}, & \nu > 1 \\ f^{(n)}_{L,a,\nu+n+1}(x), & \nu \leq -1,\quad n+\nu>-1,\quad n \in \mathbb{N}. \end{cases} \qquad (x \in \mathbb{R})$$

where n is the smallest integer such that $n+\nu>-1$.

This definition includes, for example, distribution $\delta^{(k)}(x-a)$, $a > 0$, $(H(x-a)\ln x)^{(k)}$, $a \geq 0$, $k \in \mathbb{N}$. If $a=0$ we use the notation $f_{L,\nu+1}$.

We have

$$f_{L,a,\nu+1}(x) + f_{L,b,\nu+1}(-x) \overset{q}{\sim} f_{\nu+1}(x) + f_{\nu+1}(-x), \quad x \in \mathbb{R} \quad \text{at } \pm\infty \text{ with}$$

respect to $k^{\nu}L(k)$.

Let us denote by Λ the set $\mathbb{N}$ or the set of the form $\{1,2,\dots,N\}$, $N \in \mathbb{N}$. In the second case we shall also use the symbol Λ_N. In the definition which is to follow we assume ν_i, $i \in \Lambda$, are real numbers and L_i, $i \in \Lambda$, are slowly varying functions in infinity. Also, we assume that for $i,j \in \Lambda$, $i < j$ there holds $\nu_i \geq \nu_j$ and if $\nu_i = \nu_j$, then $L_i(x)/L_j(x) \to \infty$, $x \to \infty$. Similarly as in [5]:

We say that an $f \in \mathcal{S}'$ has the quasiasymptotic expansion at $\pm\infty$ with respect to $\{(k^{\nu_i}L_i),\ i\in\Lambda\}$, if there are complex numbers $(A_i,B_i)\neq(0,0)$, (a_i,b_i), $i \in \Lambda$, such that for any $m \in \Lambda$

$$\frac{f(kx) - \sum_{i=1}^{m} A_i f_{L_i, a_i, \nu_i+1}(kx) + B_i f_{L_i, b_i, \nu_i+1}(-kx)}{k^{\nu_m} L_m(k)} \to 0 \quad \text{in } \mathcal{S}'.$$

In that case we write

(2-2) $\quad f \overset{q.e.}{\sim} \sum_{i\in\Lambda} A_i f_{L_i, a_i, \nu_i+1} + B_i f_{L_i, b_i, \nu_i+1}$ *at* ∞ *with respect to*

$\left\{(k^{\nu_i} L_i),\ i \in \Lambda\right\}.$

One can prove easily if (2-2) holds, then

$$f' \overset{q.e.}{\sim} \sum_{i\in\Lambda} A_i \left[f'_{L_i, a_i, \nu_i+1}\right] + B_i \left[f'_{L_i, b_i, \nu_i+1}\right] \text{ at } \infty \text{ with respect to}$$

$\left\{(k^{\nu_i} L_i),\ i \in \Lambda\right\}.$

Proposition 1. *Let* $f \in \mathcal{S}'$. *If* $\Lambda \subset \tilde{\Lambda}$ *and*

$f \overset{q.e.}{\sim} \sum_{i\in\Lambda} a_i f_{L_i, \nu_i+1} + b_i f_{L_i, \nu_{i+1}}$ at ∞ *with respect to* $\left\{(k^{\nu_i} L_i),\ i \in \Lambda\right\}$

and

$f \overset{q.e.}{\sim} \sum_{i\in\tilde{\Lambda}} \tilde{a}_i f_{\tilde{L}_i, \tilde{\nu}_i+1} + \tilde{b}_i f_{\tilde{L}_i, \tilde{\nu}_{i+1}}$ at ∞ *with respect to* $\left\{(k^{\tilde{\nu}_i} \tilde{L}_i),\ i \in \overset{o}{\tilde{\Lambda}}\right\}$,

then $\nu_i = \tilde{\nu}_i$, $a_i = \tilde{a}_i$, $b_i = \tilde{b}_i$ *and* $L_i(x) \sim \tilde{L}_i(x)$, $x \to \infty$ $i \in \Lambda$.

Proof. For $i=1 \in \Lambda$ we have

$$\frac{f(kx) - \left[a_1 f_{L_1, \nu_1+1}(kx) + b_1 f_{L_1, \nu_1+1}(-kx)\right]}{k_1^{\nu_1} \tilde{L}_1(k)} \to 0 \quad \text{in } \mathcal{S}' \text{ as } k \to \infty .$$

and

$$\frac{f(kx) - \left[\tilde{a}_1 f_{\tilde{L}_1, \tilde{\nu}_1+1}(kx) + \tilde{b}_1 f_{\tilde{L}_1, \tilde{\nu}_1+1}(-kx)\right]}{k_1^{\tilde{\nu}_1} \tilde{L}_1(k)} \to 0 \quad \text{in } \mathcal{S}' \text{ as } k \to \infty$$

Take $\phi \in \mathcal{D}(0,\infty)$, so that $\langle f_{\nu_1+1}(x),\phi(x)\rangle \neq 0$ and $\langle f_{\tilde{\nu}_1+1}(x),\phi(x)\rangle \neq 0$.

With the assumption $\tilde{a}_1 \neq 0$, we have

$$\left\langle \frac{f(kx) - a_1 f_{L_1,\nu_1+1}(kx)}{k_1^{\nu_1} L_1(k)}, \phi(x)\right\rangle \to 0 \quad \text{as} \quad k \to \infty$$

and

$$\frac{k^{\nu_1} L_1(k)}{k^{\tilde{\nu}_1}\tilde{L}_1(k)} \left\langle \frac{f(kx)}{k^{\nu_1} L_1(k)}, \phi(x) \right\rangle \to \tilde{a}_1 \langle f_{\tilde{\nu}_1+1}(x), \phi(x)\rangle \neq 0 \quad \text{as} \quad k \to \infty .$$

Because of

$$\lim_{k\to\infty} \left\langle \frac{f(kx)}{k^{\nu_1} L_1(k)}, \phi(x)\right\rangle = a_1 \langle f_{\nu_1+1}(x), \phi(x)\rangle ,$$

it follows that $a_1 \neq 0$ and that the limit

$$\lim_{k\to\infty} \frac{k^{\nu_1} L_1(k)}{k^{\tilde{\nu}_1}\tilde{L}_1(k)}$$

must exist and be different from zero.

This implies $\nu_1 = \tilde{\nu}_1$ and $L_1(x) \sim \tilde{L}_1(x)$ as $x \to \infty$. So $\nu_1 = \tilde{\nu}_1$, $L_1(x) \sim \tilde{L}_1(x)$ as $x \to \infty$ and thus $a_1 = \tilde{a}_1$. In the same way $b_1 = \tilde{b}_1$.

If $2 \in \Lambda$, we have

$$\frac{f(kx) - a_1 f_{L_1,\nu_1+1}(kx) - b_1 f_{L_1,\nu_1+1}(-kx) - a_2 f_{L_2,\nu_2+1}(kx) - b_2 f_{L_2,\nu_2+1}(-kx)}{k^{\nu_2} L_2(k)} \to 0$$

in $\mathcal{S}'$ as $k \to \infty$

and

$$\frac{f(kx) - a_1 f_{L_1,\nu_1+1}(kx) - b_1 f_{L_1,\nu_1+1}(-kx) - \tilde{a}_2 f_{\tilde{L}_2,\tilde{\nu}_2+1}(kx) - \tilde{b}_2 f_{\tilde{L}_2,\tilde{\nu}_2+1}(-kx)}{k^{\tilde{\nu}_2}\tilde{L}_2(k)} \to 0$$

in $\mathcal{S}'$ as $k \to \infty$.

Assume $\tilde{a}_2 \neq 0$. Take $\phi \in \mathcal{D}(0,\infty)$ so that $\langle f_{\nu_2+1},\phi(x)\rangle \neq 0$ and $\langle f_{\tilde{\nu}_2+1},\phi(x)\rangle \neq 0$.

Since,

$$\frac{k^{\nu_2}L_2(k)}{k^{\tilde\nu_2}\tilde L_2(k)}\left\langle \frac{f(kx)-a_1 f_{L_1,\nu_1+1}(kx)}{k^{\nu_2}L_2(k)},\ \phi(x)\right\rangle \longrightarrow \tilde a_2\langle f_{\tilde\nu_2+1}(x),\ \phi(x)\rangle \neq 0$$

and

$$\lim_{k\to\infty}\left\langle \frac{f(kx)-a_1 f_{L_1,\nu_1+1}(kx)}{k^{\nu_2}L_2(k)},\ \phi(x)\right\rangle = a_2\langle f_{\nu_2+1}(x),\ \phi(x)\rangle .$$

We have that the limit $\lim_{k\to\infty}\frac{k^{\nu_2}L_2(k)}{k^{\tilde\nu_2}\tilde L_2(k)}$ must exist and be different from zero.

This implies $\nu_2=\tilde\nu_2$ and $L_2(x)\sim\tilde L_2(x)$ as $x\longrightarrow\infty$ and thus $a_2=\tilde a_2$. In the same way $b_2=\tilde b_2$, $(a_2,b_2)\neq(0,0)$. The complete proof follows by induction.

As in [5] we have

Proposition 2. *(i) Let $f\in L^1_{loc}$ and $fx^n\in L^1$ for some $n\in N_0$. Then,*

$$\text{(2-3)}\quad \begin{cases} f(x) \overset{q.e}{\sim} m_0\delta(x) - m_1\delta'(x) + \dots + \dfrac{(-1)^n m_n\delta^{(n)}(x)}{n!} \text{ at } \pm\infty \text{ with} \\ \text{respect to } \{(k^{-i-1}),\ i\in\Lambda_n\} \text{ where } m_i=\displaystyle\int_{-\infty}^{+\infty} t^i f(t)\,dt,\quad i=0,1,\dots,j \end{cases}$$

(ii) Let $f\in L^1_{loc}(R\setminus\{0\})$, $A>0$ and $H(x-A)f(x)x^n + H(-x-A)f(x)|x|^n\in L^1$ for some $n\in N$. If we denote by $\tilde m_i=\int_{\infty}^{-A} t^i f(t)dt$ and $m_i=\int_{A}^{\infty} t^i f(t)dt$, $i=0,\dots,n$, then

$$\text{(2-4)}\quad \begin{cases} H(x-A)f(x) + H(-x-A)f(x) \overset{q.e.}{\sim} \displaystyle\sum_{i\in\Lambda_n}\frac{(-1)^i}{i!}\Big[m_i\delta^{(i)}(x-A) + \tilde m_i\delta^{(i)} \\ (x+A)\Big] \text{ at } \pm\infty \text{ with respect to } \{(k^{-i-1}),\ i\in\Lambda_n\} \end{cases}$$

(iii) Let $f \in L^1_{loc}(\mathbb{R}\setminus\{0\})$ *and* $\tilde{m}_n = \int_{-\infty}^{-A} t^n f(t)dt < \infty$, $m_n = \int_{A}^{\infty} t^n f(t)dt < \infty$, $n \in \mathbb{N}_0$, $A > 0$. *Then,*

$$H(x-A)f(x) + H(-x-A)f(x) \overset{q.e.}{\sim} \sum_{i\in A_{\mathbb{N}}} \frac{(-1)^i}{i!}\left[m_i\delta^i(x-A) + \tilde{m}_i\delta^i(x+A)\right]$$

at $\pm\infty$ *with respect to* $\{(k^{-i-1}),\ i \in \mathbb{N}\}$.

(iv) Let $h \in L^1_{loc}$ *and* $h(x) \sim C_1/x^n$, $x \to \infty$, $h(x) \sim C_2/|x|^n$, $x \to -\infty$ $n \in \mathbb{N}$, $n \geq 2$.

Then

$$\text{(2-5)}\quad \begin{cases} h(x) \overset{q.e.}{\sim} C_1\Big[m_0\delta(x) - m_1\delta'(x) + \ldots + (-1)^{n-2}m_{n-2}\delta^{(n-2)}(x) + \\ \qquad + \dfrac{(-1)^{n-1}}{(n-1)!}(H(x)\ln x)^{(n)}\Big] + \\ \qquad + C_2\Big[\tilde{m}_0\delta(x) - \tilde{m}_1\delta'(x) + \ldots + (-1)^{n-2}\tilde{m}_{n-2}\delta^{(n-2)}(x) + \\ \qquad + \dfrac{(-1)^{n-1}}{(n-1)!}(H(-x)\ln|x|)^{(n)}\Big] \end{cases}$$

at $\pm\infty$ with respect to $\{(k^{-1}L_i), i \in A_n\}$, where $L_i = 1$, $i=1,\ldots,n-1$, $L_n(x) = \ln|x|$ and $m_i = \int_0^{\infty} \frac{t^i h(t)}{i!}dt$, $\tilde{m}_i = \int_{-\infty}^{0} \frac{t^i h(t)}{i!}dt$, $i=0,\ldots,n-2$.

We omit the proof.

We also need from [5]

$$\text{(2-6)}\qquad \frac{H(x-1)}{x^n} = \frac{1}{(n-1)!}\left(\sum_{j=0}^{n-2}(-1)^j(n-2-j)!\,\delta^{(j)}(x-1) + \right.$$
$$\left. + (-1)^{n-1}(H(x-1)\ln x)^{(n)}\right) .$$

Remark. The quasiasymptotic expansion is not unique (see [5]).

3. Applications

We shall give the applications for the classical Stieltjes transform. Namely, we shall give assertions related to the classical transform to emphasize the fact that we obtain new "classical" results by using the abstract theory from the preceding section.

We shall follow the definition of the distributional Stieltjes transform given by Lavoine and Misra.

Let $f \in \mathcal{S}'$. We say that $f \in J'(r)$, if there exist $m \in \mathbb{N}_0$ and a locally integrable function F such that

$$\text{(3-1)} \quad \begin{cases} \text{a) } f = F^{(m)} \\ \text{b) } \displaystyle\int_{-\infty}^{\infty} |F(x)(x+z)^{-r-m-1}|dx < \infty \text{ for } Im(z) \neq 0 . \end{cases}$$

The Stieltjes transform S_r of index r, $r \in \mathbb{R}\setminus(-\mathbb{N}_0)$ of a distribution $f \in J'(r)$ with the properties given in (3-1) is a complex valued function given by

$$\text{(3-2)} \quad (S_r f)(z) = (r+1)_m \int_{-\infty}^{\infty} F(x)(x+z)^{-r-m-1}dx = (r+1)_m \langle F(x), \frac{1}{(x+z)^{r+m+1}}\rangle, Im(z) \neq 0,$$

where $(r)_k = r(r+1)\ldots(r+k-1)$, $k > 0$ *and* $(r)_0 = 1$.

It is easy to see that $(S_r f)(z)$ is a holomorphic function of the complex variable z in the domain $\mathbb{C}\setminus(-\infty,+\infty)$.

We need here the following result which is a modification of the Abelian theorem from [3].

Abelian theorem. Let $f \in \mathcal{S}'$ and $f(kx)/k^{\nu}L(k) \to 0$ in $\mathcal{S}'$, $k \to \infty$, $\nu \in \mathbb{R}\setminus(-\mathbb{N})$. Let $r \in \mathbb{R}\setminus(-\mathbb{N})$ and $r > \nu$. Then, for any $z \in \mathbb{C}\setminus\mathbb{R}$

$$\lim_{k\to\infty} \frac{(S_r f)(kz)}{k^{\nu-r}L(k)} = 0.$$

The following formulas are needed

$$\text{(3-3)} \qquad (r+1)_p (S_{r+p} f)(z) = (S_r f^{(p)})(z), \quad f \in J'(r+p), \ z \in \mathbb{C}\setminus(-\infty,+\infty)$$

$$\text{(3-4)} \qquad S_r(\delta^{(k)}(x-A))(z) = \frac{(r+1)_k}{(z+A)^{r+k+1}}, \ r > -k-1, \ A \ge 0 .$$

(3-5) $$S_r(\delta^{(k)}(x+A))(z) = \frac{(r+1)_k}{(z-A)^{r+k+1}},\ r > -k-1,\ A \geq 0\ .$$

According to Proposition 2, (2-6), the Abelian theorem and (3-3), (3-4), (3-5), we obtain

Proposition 3. *(i) Let $f \in L^1_{loc}$ and $fx^n \in L^1$ for some $n \in \mathbb{N}_0$. Then, for $r>-1$, and $z \in \mathbb{C}\setminus\mathbb{R}$, with $m_i = \int_{-\infty}^{\infty}\frac{t^i f(t)dt}{i!}$ $i=0,\dots,n$,*

$$\frac{(S_r f)(kz) - m_0\frac{1}{(kz)^{r+1}} + m_1\frac{r+1}{(kz)^{r+2}} - \dots - (-1)^n m_n\frac{(r+1)^n}{(kz)^{r+n+1}}}{k^{-r-n-1}} \longrightarrow 0,\ k \longrightarrow \infty.$$

(ii) Let $f \in L^1_{loc}(\mathbb{R}\setminus\{0\})$, $A > 0$, $H(x-A)f(x)x^n + H(-x-A)f(x)|x|^n \in L^1$ for some $n \in \mathbb{N}$. If we denote $\tilde{m}_i = \int_{-\infty}^{-A} t^i f(t)dt$, $m_i = \int_{A}^{\infty} t^i f(t)dt$, $i=0,\dots,n$ and if $r>-1$, then

$$\frac{S_r(H(t-A)f(t))(kz) + S_r(H(-t-A)f(t))(kz)}{k^{-n-r-1}} -$$

$$- \frac{\sum_{i=0}^{n}\frac{(-1)^i}{i!}(r+1)_i\left[\frac{m_i}{(kz+A)^{r+i+1}} + \frac{\tilde{m}_i}{(kz-A)^{r+i+1}}\right]}{k^{-n-r-1}} \longrightarrow 0$$

as $k \longrightarrow \infty$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$.

(iii) Let $h \in L^1_{loc}$, $h(x) \sim C_1/x^n$, $x \longrightarrow \infty$, $h(x) \sim C_2/|x|^n$, $x \longrightarrow -\infty$, $n \geq 2$, $n \in \mathbb{N}$, ($C_1 \neq 0$, $C_2 \neq 0$).

Then, for $r > -1$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$,

$$(3\text{-}6)\quad \begin{cases} (S_r h)(kz) = \dfrac{1}{(kz)^{r+1}}(m_0C_1 + \tilde{m}_0C_2) - \dfrac{r+1}{(kz)^{r+2}}(m_1C_1 + \tilde{m}_1C_2)+\ldots+ \\ \qquad + (-1)^{n-2}\dfrac{(r+1)_{n-2}}{(kz)^{r+n-1}}(m_{n-2}C_1 + \tilde{m}_{n-2}C_2) + \\ +\dfrac{(-1)^{n-1}}{(n-1)!}(r+1)_n\left[C_1\left(\dfrac{A_0(z)}{k^{r+n}}+\dfrac{\ln k}{k^{r+n}}\,\dfrac{1}{(r+n)z^{r+n}}\right)+C_2\left(\dfrac{A_1(z)}{k^{r+n}}-\dfrac{\ln k}{k^{r+n}}\,\dfrac{1}{(r+n)z^{r+n}}\right)\right] \\ + O_z(k^{-n-r}\ln k) \quad \text{as} \quad k\to\infty, \end{cases}$$

where $m_i = \displaystyle\int_0^\infty \frac{t^i h(t)}{i!}\,dt$, $\tilde{m}_i = \displaystyle\int_{-\infty}^0 \frac{t^i h(t)}{i!}\,dt$, $i=0,\ldots,n-2$,

$A_0(z) = \displaystyle\int_0^\infty \frac{\ln t}{(t+z)^{r+n+1}}\,dt$, $A_1(z) = \displaystyle\int_0^\infty \frac{\ln t}{(-t+z)^{r+n+1}}\,dt$, $O_z(k^{-n-r}\ln k)$ *is a function which depends on* z *and*

$$\frac{O_z(k^{-n-r}\ln k)}{k^{-n-r}\ln k} \longrightarrow 0, \quad k\longrightarrow\infty.$$

(iv) For $n \ge 2$, $r > -1$, $r \notin \mathbb{N}_0$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$, $|z| > 1$, $k > 0$,

$$(3\text{-}7)\qquad \left[S_r\,\frac{H(t-1)}{t^n}\right](kz) = \frac{1}{(n-1)!}\sum_{j=0}^{n-2}(-1)^j(n-2-j)!\,\frac{(r+1)_j}{(kz+1)^{r+j+1}}$$

$$+\frac{(-1)^{n-1}(r+1)_n}{k^{r+n}}\left[A_2(k,z) + A_3(z) + \frac{\ln k}{z^{r+n}}\right]$$

where $A_2(k,z) = \dfrac{1}{(\alpha)_{[r+n]}}\displaystyle\sum_{p=1}^{\infty}\binom{-\alpha}{p}\left(1-\frac{1}{k^p}\right)\frac{(\alpha+p)_{[r+n]}}{pz^{r+n+p}}$, $\alpha = r+n-[r+n]$ *and*

$$A_3(z) = \int_1^\infty \frac{\ln t}{(z+t)^{r+n+1}}\,dt\ .$$

Note that for $p \in \mathbb{N}_0$, $\left[S_r \frac{H(t-1)}{t^n}\right](kz)$ has a simpler form.

Proof. We shall prove only (iii) and (iv), because (i) and (ii) directly follow from Proposition 2.

(iii) For $r > -1$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$,

$$\lim_{k\to\infty}\left[\frac{(S_r h)(kz) - \frac{1}{(kz)^{r+1}}(m_0 C_1 + \bar{m}_0 C_2) + \frac{r+1}{(kz)^{r+2}}(m_1 C_1 + \bar{m}_1 C_2) +}{k^{-n-r}\ln(k)} + \ldots +\right.$$

$$\frac{+(-1)^{n-2}\frac{(r+1)_{n-2}}{(kz)^{r+n-1}}(m_n C_1 + \bar{m}_n C_2)}{k^{-n-r}\ln(k)} - \frac{\frac{(-1)^{n-1}}{(n-1)!}(r+1)_n}{k^{-n-r}\ln(k)}$$

$$\left.\cdot\left[C_1\int_0^\infty \frac{\ln t}{(t+kz)^{r+n+1}}\,dt + C_2\int_{-\infty}^0 \frac{\ln|t|}{(t+kz)^{r+n+1}}\,dt\right]\right] = 0 .$$

Since

$$\int_0^\infty \frac{\ln t}{(t+kz)^{r+n+1}}\,dt = \frac{1}{k^{r+n}}A_0(z) + \frac{\ln k}{k^{r+n}}\frac{1}{(r+n)z^{r+n}}$$

and

$$\int_{-\infty}^0 \frac{\ln|t|}{(t+kz)^{r+n+1}}\,dt = \int_0^\infty \frac{\ln t}{(-t+kz)^{r+n+1}}\,dt = \frac{1}{k^{r+n}}A_1(z) - \frac{\ln k}{k^{r+n}}\frac{1}{(r+n)z^{r+n}} .$$

we get for $z \in \mathbb{C}\setminus(-\infty,+\infty)$ (3-6).

(iv) For $n \ge 2$, $r > -1$, $r \notin \mathbb{N}_0$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$ we have

$$\left(S_r\frac{H(t-1)}{t^n}\right)(z) = \frac{1}{(n-1)!}\left[\sum_{j=0}^{n-2}(-1)^j(n-2-j)!\frac{(r+1)_j}{(z+1)^{r+j+1}} + (-1)^{n-1}(r+1)_n\int_1^\infty \frac{\ln t}{(z+t)^{r+n+1}}\,dt\right].$$

If $n=1$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$

$$(S_r \frac{H(t-1)}{t^1})(z) = (r+1)\int_1^{\infty} \frac{\ln t}{(z+t)^{r+2}}\,dt\,.$$

So, we have ($n \geq 2$)

$$(S_r \frac{H(t-1)}{t^n})(kz) = \frac{1}{(n-1)!}\left[\sum_{j=0}^{n-2}(-1)^j(n-2-j)!\frac{(r+1)_j}{(kz+1)^{r+j+1}} + (-1)^{n-1}(r+1)_n\int_1^{\infty}\frac{\ln t}{(t+kz)^{r+n+1}}dt\right]$$

Let us compute the last integral. Assume that $|z| > 1$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$.
We have

$$J(z) = \int_1^{\infty}\frac{\ln t}{(t+kz)^{r+n+1}}\,dt = \frac{1}{k^{r+n}}\left(\int_{1/k}^{\infty}\frac{\ln u}{(z+u)^{r+n+1}}\,du + \ln k\int_{1/k}^{\infty}\frac{du}{(z+u)^{r+n+1}}\right] =$$

$$= \frac{1}{k^{r+n}}\left(\left[\int_{1/k}^{1}\frac{\ln u}{(z+u)^{r+n+1}}\,du + \int_1^{\infty}\frac{\ln u}{(z+u)^{r+n+1}}\,du + \frac{1}{(r+n)}\frac{1}{(z+1/k)^{r+n}}\right]\right.\,.$$

Since

$$\int_{1/k}^{1}\frac{\ln u}{(z+u)^{r+n+1}}\,du = \frac{-\ln k}{(r+n)}\frac{1}{(z+1/k)^{r+n}} + \int_{1/k}^{1}\frac{du}{u(z+u)^{r+n}}\,du,$$

we get

$$J(z) = \frac{1}{k^{r+n}}\left[\int_{1/k}^{1}\frac{du}{u(z+u)^{r+n}} + \int_1^{\infty}\frac{\ln u}{(z+u)^{r+n+1}}\,du\right]$$

Let $\alpha = r+n-[r+n]$. Because of $(z+u)^{-\alpha} = z^{-\alpha}\left[1+\frac{u}{z}\right]^{-\alpha} = z^{-\alpha}\sum_{p=0}^{\infty}\binom{-\alpha}{p}\frac{u^p}{z^p}$,

we have

$$J_{\alpha}(z) = \int_{1/k}^{1}\frac{du}{u(z+u)^{\alpha}} = \sum_{p=1}^{\infty}(1-1/k^p)\binom{-\alpha}{\mu}\frac{1}{pz^{\alpha+p}} + \frac{\ln k}{z^{\alpha}}$$

So, by differentiating the last equality $[r+n]$- times, we get

Note that for $p \in \mathbb{N}_0$, $\left[S_r \dfrac{H(t-1)}{t^n}\right](kz)$ has a simpler form.

Proof. We shall prove only (iii) and (iv), because (i) and (ii) directly follow from Proposition 2.

(iii) For $r > -1$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$,

$$\lim_{k\to\infty}\left[\frac{(S_r h)(kz) - \frac{1}{(kz)^{r+1}}(m_0 C_1 + \tilde{m}_0 C_2) + \frac{r+1}{(kz)^{r+2}}(m_1 C_1 + \tilde{m}_1 C_2) +}{k^{-n-r}\ln(k)} + \dots + \right.$$

$$\frac{+(-1)^{n-2}\frac{(r+1)_{n-2}}{(kz)^{r+n-1}}(m_n C_1 + \tilde{m}_n C_2)}{k^{-n-r}\ln(k)} - \frac{\frac{(-1)^{n-1}}{(n-1)!}(r+1)_n}{k^{-n-r}\ln(k)}$$

$$\left.\cdot\left[C_1\int_0^\infty \frac{\ln t}{(t+kz)^{r+n+1}}\,dt + C_2\int_{-\infty}^0 \frac{\ln|t|}{(t+kz)^{r+n+1}}\,dt\right]\right] = 0\ .$$

Since

$$\int_0^\infty \frac{\ln t}{(t+kz)^{r+n+1}}\,dt = \frac{1}{k^{r+n}}A_0(z) + \frac{\ln k}{k^{r+n}}\frac{1}{(r+n)z^{r+n}}$$

and

$$\int_{-\infty}^0 \frac{\ln|t|}{(t+kz)^{r+n+1}}\,dt = \int_0^\infty \frac{\ln t}{(-t+kz)^{r+n+1}}\,dt = \frac{1}{k^{r+n}}A_1(z) - \frac{\ln k}{k^{r+n}}\frac{1}{(r+n)z^{r+n}}\ ,$$

we get for $z \in \mathbb{C}\setminus(-\infty,+\infty)$ (3-6).

(iv) For $n \ge 2$, $r > -1$, $r \notin \mathbb{N}_0$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$ we have

$$\left(S_r\frac{H(t-1)}{t^n}\right)(z) = \frac{1}{(n-1)!}\left[\sum_{j=0}^{n-2}(-1)^j(n-2-j)!\frac{(r+1)_j}{(z+1)^{r+j+1}} + (-1)^{n-1}(r+1)_n\int_1^\infty\frac{\ln t}{(z+t)^{r+n+1}}\,dt\right].$$

If $n=1$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$

$$J_{\alpha}^{([r+n])}(z) = (-1)^{[r+n]}\alpha(\alpha+1)\dots(r+n-1)\int_{1/k}^{1}\frac{du}{u(z+u)^{r+n}} .$$

This implies

$$\int_{1/k}^{1}\frac{du}{n(z+u)^{r+n}} - \frac{\ln k}{z^{r+n}} =$$

$$= \frac{1}{(-1)^{[r+n]}\alpha(\alpha+1)\dots(r+n-1)}\sum_{p=1}^{\infty}(1-1/k^{p})\begin{bmatrix}-\alpha\\ p\end{bmatrix}\frac{(-1)^{[r+n]}(\alpha+p)\dots(r+n+p+1)}{pz^{r+n}}$$

i.e.

$$\int_{1/k}^{1}\frac{du}{u(z+u)^{r+n}} = A_2(k,z) + \frac{\ln k}{z^{r+n}} ,$$

where $A_2(k,z) = \dfrac{1}{\alpha(\alpha+1)\dots(r+n-1)}\displaystyle\sum_{p=1}^{\infty}\begin{bmatrix}-\alpha\\ p\end{bmatrix}(1-1/k^{p})\frac{(\alpha+p)\dots(r+n+p-1)}{z^{r+n+p}}.$

If we denote by $A_3(z) = \displaystyle\int_{1}^{\infty}\frac{\ln t}{(z+t)^{r+n+1}}\,dt,$ we obtain

$J(z) = \dfrac{1}{k^{r+n}}\left(A_2(k,z) + \ln k/z^{r+n} + A_3(z)\right)$ and formula (3-7).

By using formulas (3-6) and (3-7), one can get the classical result for the asimptotic expansion of the Stieltjes transform of a function which has the classical asymptotic expansion

$$f(t) \sim \sum_{i=1}^{\infty}\frac{C_i}{t^i}, \quad t \to \infty$$

Namely, for any $n \in \mathbb{N}$ we have

$$f(t) - \sum_{i=1}^{n-1}\frac{C_i H(t-1)}{t^i} \sim \frac{C_n}{t^n}, \quad t \to \infty$$

and by using the quoted formulas we obtain the asymptotic expansion of $(S_r f)(kz)$ on the ray kz, $k \to \infty$, $z \in \mathbb{C}\setminus(-\infty,+\infty)$.

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Rezime

KVAZIASIMPTOTSKI RAZVOJ TEMPERIRANIH DISTRIBUCIJA
I STIELTJESOVA TRANSFORMACIJA

Definisan je kvaziasimptotski razvoj temperiranih distribucija na realnoj pravoj i dokazane su neke osobine ovog razvoja, Tvrdenja 1. i 2. Takode, data je primena kvaziasimptotskog razvoja na distribucionu Stieltjesovu transformaciju, Tvrdenje 3.

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