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A CLASSIFICATION OF INTERVAL GREEDOIDS ON AT MOST 5 ELEMENTS

Dragan M. Acketa

Institute of Mathematics, Dr Ilije Djuricica 4, 21000 Novi Sad, Yugoslavia

Abstract

The paper includes a construction of all the non-isomorphic greedoids on at most 5 elements in five classes of interval greedoids: local poset greedoids, poset greedoids, undirected branching greedoids, directed branching greedoids and convex shellings. A computer-aided search on the formerly generated ([3]) catalogue of all the non-isomorphic interval greedoids on at most 5 elements was performed for the extraction of the greedoids in the first two classes. The remaining three classes were constructed "by hand", on the basis of some more general theoretical considerations (which are not restricted to the sets consisting of at most 5 elements).

1. Introduction

An *n-set* is a set of cardinality n. Sets are often denoted without brackets and commas.

Greedoid ([5]) G on a finite set (the *ground-set*) E is an ordered pair (E,F), where F is a family of so-called *feasible* subsets of E which satisfies:

(i) $\emptyset \in F$

(ii) $(\forall X \in F)\ (\exists e \in X)\ (X-e \in F)$

(iii) if $(X,Y \in F)$ and $(|X|=|Y|+1)$ then $(\exists y \in Y)\ (Y \cup x \in F)$

A greedoid (E,F) is *interval* if it satisfies additionally:

if $(A \subset B \subset C \subset E)$ and $(x \in E-C)$ and $(A,\ B,\ C,\ A \cup x,\ C \cup x \in F)$ then $B \cup x \in F$

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A greedoid (E, F) is *full* if $E \in F$. A greedoid (E, F) is *normal* if each element from E belongs to a set from F.

The *rank* of a greedoid is the maximal cardinality of a feasible set.

We shall proceed with the definitions of seven subclasses of interval greedoids (E, F). The name of each class is followed by its recognizing property:

1) *Local poset greedoids*: If $A, B, C \in F$ and $A, B \subseteq C$, then $A \cup B, A \cap B \in F$.

2) *Poset greedoids*: If $A, B \in F$, then $A \cup B, A \cap B \in F$.

3) *Undirected branching greedoids* (UBG's): Let E be the edge-set of an undirected graph G and let r (= *root*) denote a fixed vertex of G. Then $F = \{ X \subseteq E \mid X$ is a subtree of G containing $r \}$.

4) *Directed branching greedoids* (DBG's): The only differences from UBG's: "undirected" and "subtree" are respectively replaced by "directed" and "arborescence = a subtree directed from the root".

5) *Convex shellings*: Let E be a finite set of vectors in R^n. Then $F = \{ \{x(1),\ldots,x(k)\} \mid x(i)$ is a vertex of the convex hull of $E - \{x(1),\ldots,x(i-1)\}$ for $1 \le i \le k \}$.

6) *Matroids:* *(ii)* is replaced by $X \in F,\ Y \subseteq X \Rightarrow Y \in F$.

7) *Shelling structures:* $E \in F$.
(this definition is valid only if the interval property is separately assumed - intervality is a consequence of the former properties)

2. Local poset greedoids

An algorithm for testing the local poset property is given by the following procedure (written in pseudoPascal, as well as the other algorithms described below):

Given a greedoid (E,F)

```
REPEAT
   Take the next unordered pair {A,B} of different subsets of E
   IF (A in F) THEN
      IF (B in F) THEN BEGIN
         Un: = union (A, B);
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        Is: = intersection (A, B);
        IF not (Un in F) or not (Is in F) THEN
           IF Un in F THEN stop
           ELSE REPEAT
                     X: = the next superset of Un;
                               (within E)
                     IF X in F THEN stop
                 UNTIL stop or (all supersets of
                          Un within E are examined)
    END
UNTIL stop or all the possible pairs {A,B} are examined;

IF stop then the greedoid is not local poset
ELSE it is local poset
```

The point of this algorithm is that attention is restricted solely to those pairs $\{A, B\}$, which contradict the poset property. The local poset property is then also contradicted if and only if there exists a feasible set including the union (A, B).

The ground-set E is represented by $\{1,...n\}$, for $1 \le n \le 5$.

The "small" interval greedoids obtained in [3], which are subject to the local poset test, are represented by their families of feasible sets. These sets are represented as binary (incidence) vectors.

The following table contains the number of non-isomorphic local poset greedoids of rank r on an n-set, for $0 \le r \le n \le 5$:

n	0	1	2	3	4	5	r
	1	1	1	1	1	1	0
		1	2	3	4	5	1
			2	8	25	70	2
				5	44	411	3
					16	356	4
						63	5

3. Poset greedoids

A relaxation of the previous procedure can be applied to test the poset property (the third IF..THEN should immediately imply stop). Another easy way to obtain poset greedoids from local poset greedoids is to use the fact that normal poset greedoids = full local poset greedoids .

Thus, the figures on the main diagonal of the previous table correspond to normal poset greedoids.

Note, however, that extracting the poset greedoids from general greedoids is by no means an advisable way for poset enumeration; it is hopeless to try this for $|E| > 5$.

The last three classes are directly constructed "by hand". We produce solely normal branching greedoids (undirected and directed); the non normal greedoids on k elements are obtained by bijection to the greedoids on smaller ground-sets (= edge-sets). Thus, the total number of greedoids in these classes is obtained by summing up the number of normal greedoids from 0 to k.

4. Undirected branching greedoids

We shall start with a list of all the non-isomorphic connected loopless undirected multigraphs on at most 5 edges (parallel edges are allowed; the simple graphs in the considered list can be found, e.g., in [4]). Given a graph G in this list, we choose the corresponding root r in all the non-isomorphic ways (i.e., the chosen roots are the representatives of the orbits of the automorphism group on the vertex-set). Each pair (G,r) corresponds to an undirected branching greedoid. It is obvious that all these greedoids are normal; each edge belongs to a rooted subtree due to the connectedness and to the absence of loops.

We shall list the considered graphs G (the graphs are given by the list of their edges) as well as the collections of their corresponding roots in mutually non-isomorphic positions:

LIST OF UNDIRECTED BRANCHING GREEDOIDS

Graph	Corresponding roots
G 1 = ø.	
G 2 = ab.	a
G 3 = ab, ac.	a, b
G 4 = 2ab.	a.
G 5 = ab, ac, bd.	a, c.
G 6 = ab, ac, bc.	a.
G 7 = ab, ac, ad.	a, b
G 8 = 3ab.	a.
G 9 = 2ab, ac	a, b, c.
G 10 = ab, ac, bd, ce.	a, b, d.
G 11 = ab, ac, ad, bc.	a, b, c, e.
G 12 = ab, ac, ad, ae.	a, b.
G 13 = ab, ac, ad, bc.	a, b, d.
G 14 = ab, ac, bd, cd.	a.
G 15 = 4ab.	a.
G 16 = 3ab, ac.	a, b, c.

LIST OF UNDIRECTED BRANCHING GREEDOIDS
(continued)

G 17 = 2ab, 2ac.	a, b.
G 18 = 2ab, ac, cd.	a, b, c, d.
G 19 = 2ab, ac, bd.	a, c.
G 20 = 2ab, ac, ad.	a, b, c.
G 21 = 2ab, ac, bc.	a, c.
G 22 = ab, ac, bd, ce, df.	a, c, e.
G 23 = ab, ac, ad, be, ef.	a, b, c, e, f.
G 24 = ab, ac, ad, ae, bf.	a, b, c, f.
G 25 = ab, ac, ad, be, cf.	a, b, d, e.
G 26 = ab, ac, ad, be, bf.	a, c.
G 27 = ab, ac, ad, ae, af.	a, b.
G 28 = ab, ac, ad, bc, de.	a, b, d, e.
G 29 = ab, ac, ad, ae, bc.	a, b, d.
G 30 = ab, ac, ad, bc, be.	a, c, d.
G 31 = ab, ac, ad, be, ce.	a, b, d, e.
G 32 = ab, ac, bd, ce, de.	a.
G 33 = ab, ac, ad, bc, bd.	a, c.
G 34 = 5ab	a.
G 35 = 4ab, ac.	a, b, c.
G 36 = 3ab, 2ac.	a, b, c.
G 37 = 3ab, ac, cd.	a, b, c, d.
G 38 = 3ab, ac, bd.	a, c.
G 39 = 2ab, 2ac, bd.	a, b, c, d.
G 40 = 2ab, 2cd, ac.	a, b.
G 41 = 3ab, ac, ad.	a, b, c.
G 42 = 2ab, 2ac, ad.	a, b, d.
G 43 = 3ab, ac, bc.	a, c.
G 44 = 2ab, 2ac, bc.	a, b.
G 45 = 2ab, ac, cd, de.	a, b, c, d, e.
G 46 = 2ab, ac, bd, de.	a, b, c, d, e.
G 47 = 2ab, ac, ad, ce.	a, b, c, d, e.
G 48 = 2ab, ac, ad, be.	a, b, c, e.
G 49 = 2ab, ab, bd, be.	a, b, c, d.
G 50 = 2ab, ac, ad, ae.	a, b, c.
G 51 = 2ab, ab, ac, ad.	a, b, d.
G 52 = 2ab, ac, ad, bc.	a, b, c, d.
G 53 = 2ab, ac, ad, cd.	a, b, c.
G 54 = 2ab, ac, bd, cd.	a, c.

The following reconstruction lemma guarantees that all the UBC's contained in the above list are pairwise non-isomorphic:

Lemma 4.1. *Given an undirected branching gredoid (E,F), the corresponding rooted (connected loopless) undirected graph (G, r) can be uniquely reconstructed (not only up to an isomorphism, but also up to the denotations of elements (edges)).*

Before proving this lemma, we shall introduce a couple of definitions:

Given a vertex *r* of a connected graph *G*, another vertex *v* of *G* is said to be at a *distance* *k* from *r* if the shortest *rv*-path contains *k* edges.

Similarly, an edge is said to be at a *distance k* from r if either both of its vertices are at a *distance k-1*, or one vertex (the inner) is at a *distance k-1*, while the other (the outer) is at a distance k (from r).

A *generalized edge* is a maximal bundle of mutually parallel edges (possibly containing only one edge). It is obvious that all the edges of a generalized edge are at the same distance from r. An *outer star at distance k* (from r) is a set St of generalized edges at a distance k having a common outer vertex (if $|St|=1$, then the outer star is just a generalized edge).

Proof of Lemma 4.1 : We can sketch an algorithm for the reconstruction:

```
k: = 0 ;
REPEAT
     k: = k+1 ;
     Extract the edges of G  which are at  distance k  from the root
     r; they  are recognized as  the elements of E which appear in a
     k-set of F, but not in a (k-1)-set of F;
     FOR each two edges x and y at distance k from r DO
          IF the set {x,y} is not included into a set S of F
          (where |S| >= k+1) THEN
               the edges x and y are parallel
               (they belong to a bundle of parallel edges
               at distance k from r);
          (in this FOR-loop we have completed the set of generalized
           edges, which are at a distance k from r)
     FOR each generalized edge X at distance k from r DO
          IF  k=1   THEN
                    Make the inner vertex of X equal to r
             ELSE   BEGIN
                    Determine the number n(X) of outer stars St
                    at distance k-1 which satisfy:
                        There exist x ∈ X and s ∈ St such that the
                        the set {x, s} is included into a k-set of F;
                     (the number n(X) belongs to the set {1,2};
                     it is >=1 due to (ii) and it is <=2 due to the
                     fact that an edge has only two endpoints )
                    IF n(X)=2 THEN
                         Make the vertices of X equal to the outer
                         vertices of the corresponding stars St1 and
                         St2
```

ELSE (if $n(X)=1$)
Make the inner vertex of X equal to the (outer) vertex of St
END; (else)
IF $k > 1$ THEN
FOR each two generalized edges X and Y at distance k from r, which satisfy
$n(X) = n(Y) = 1$ DO BEGIN
Let $x \in X$ and $y \in Y$; (an arbitrary choice)
IF the set $\{x,y\}$ belongs to a (k+1)-set of F THEN
Make the outer vertices of X and Y coincide (equivalently, put X and Y into the same outer star at distance k)
END (for)
(in this FOR-loop we have completed the set of outer stars of G which are at a distance k)
UNTIL all the elements of E are exhausted (= extracted). #

Remark. Each case $n(X)=2$, as well as each pair $\{X,Y\}$ in the last FOR-loop, corresponds to the completion of a new generalized cycle (consisting of generalized edges).

(see also the remarks after the proof of Lemma 5.1)

We shall state a fact which is useful for an effective production of UBG's from undirected rooted graphs:

Lemma 4.2 *Given an undirected graph G, all the undirected branching greedoids (G,r) w.r.t. different roots r, have the same family of bases (= maximal feasible sets = feasible sets of the maximal cardinality).*

Proof. These bases correspond to the spanning trees of G, which are known to be uniquely determined. #

5. Directed branching greedoids

An edge oriented from x to y is denoted by (xy).

An *orientation* of a graph is a collection of orientations of its edges.

A reconstruction lemma similar to Lemma 4.1. holds for the directed case, too. Such a lemma again enables the construction of the lists of non--isomorphic branching greedoids by using the lists of non-isomorphic corresponding rooted graphs:

Lemma 5.1 *Given a directed branching greedoid (E, F), the corresponding rooted (connected loopless) directed graph (G, r) can be uniquely reconstructed (non only up to an isomorphism, but also up to the denotations of elements (edges)).*

Proof. A reconstruction algorithm similar to the one given in the proof of Lemma 4.1 can be applied. We point out solely the differences from the undirected case:

- The word "edge" is throughout the algorithm replaced by "oriented edge". Similarly, the word "path" (in the definition of distance) is replaced by "oriented path" (thus we can speak about "oriented distance") and "generalized edge" is replaced by "generalized oriented edge". It is understood that all the mutually parallel edges within the same generalized oriented edge are oriented in the same direction.

- We need not determine the number $n(X)$, since it must be equal to 1; one edge (also generalized edge) cannot be oriented in two opposite directions at the same time.

- An oriented edge necessarily has an inner and an outer vertex (the *source* and the *sink*, respectively).

- We cannot speak about "the distance of an outer star" in the non--oriented sense. The oriented edges having the common outer vertex need not be at the same oriented distance from the root; it is convenient to declare the distance of an outer star equal to the maximal oriented distance of the corresponding oriented edges.

- The test for completing the outer stars becomes more complicated. Given a generalized oriented edge X at a distance k, let $f(X)$ denote an arbitrarily chosen k-set in F, which includes one oriented edge from X (in other words, $f(X)$ is an oriented path of length k from the root, which intersects X). When running our test, we also need the natural extension of the function f to the case when X is an outer star. We give the oriented version of this test:

  IF $k > 1$ THEN BEGIN

  FOR each two generalized oriented edges X and Y at distance k from r DO

IF $f(X) \cup f(Y)$ is not in F THEN
Make the outer vertices of X and Y coincide;
(i.e., make X and Y belong to the same outer star at distance k)
FOR each outer star X at distance k from r DO
(X is generated in the previous FOR-loop, possibly X is just a generalized oriented edge)
FOR each outer star Y at a distance smaller than k DO
IF $f(X) \cup f(Y)$ is not in F THEN
Make the vertices corresponding to X and Y coincide
(In this way we possibly augment the outer stars at distance k. After passing through this double FOR-loop, we have completed a temporary set of outer stars at distance k. However, we must note that some of these stars may later be fused together with an outer star at a distance greater than k)
END; (if) #

We shall give some remarks related to the reconstruction algorithms described in the proofs of Lemmas 4.1 and 5.1:

a) There is a 1-1 correspondence between the (final) outer stars and those vertices v of the reconstructed graphs, which have the property that there is an oriented path of a length greater than 1 from r to v.

b) A lot of sets in F may not be taken into account when running our algorithms. For example, given the UBG $\{\emptyset$, a, b, ac, bc, bd, acd, bcd$\}$, we do not consider the feasible 3-sets when reconstructing the graph. However, the absence of the sets acd and bcd would mean a contradiction w.r.t. our reconstructed graph. An elimination idea based on the absent, but compulsory, feasible sets - might be used for testing whether a general interval greedoid is UBG (respectively DBG).

c) Some steps can be shortened. For example, if we find that $\{X,Y\}$ and $\{Y, Z\}$ are pairs of mutually parallel edges (or, similarly, pairs of generalized oriented edges in the same outer star) - then we need not check the same thing for the edge $\{X, Z\}$. However, we need not discuss (and try to improve) the efficiency of the described algorithms; their existence is sufficient for our purposes.

We use for input the same pairs (G, r) as in the previous section. Given a pair (G, r), we consider solely those orientations of edges which make each vertex of G reachable from the root. Then, the directed branching greedoids are associated to the triples consisting of the underlying undirected graph G, the root r and the corresponding "reachable" orientations. This consideration is simplified by the application of the following lemma:

Lemma 5.2. *Given a connected loopless undirected graph G with no cycles of a length greater than 2 and a vertex r of G, there exists a unique orientation of edges of G such that the directed branching greedoid associated to the pair (G,r) is normal. This greedoid coincides with the corresponding undirected branching greedoid.*

Proof. The graph G may be considered as a generalized tree; some of its edges are possibly replaced by multiple edges. The mentioned orientation is obtained by orienting each edge FROM the root r. It is then obvious that the feasible sets (= subtrees rooted at r) are the same, both in the directed and in the undirected case.

The following two lemmas are a supplement to the previous one:

Lemma 5.3. *Given a connected loopless undirected graph G, which has a cycle of a length greater than 2, and a vertex r of G, - none of the directed branching greedoids associated to the pair (G,r), w.r.t. different orientations of edges - is isomorphic to the corresponding undirected branching greedoid.*

Proof. It is obvious that each feasible set belonging to a DBG also belongs to the UBG (it suffices to cancel the orientation).

On the other hand, consider a cycle C of a length greater than 2. There exists a vertex v of C, which either coincides with the root r or is connected to the root by a path P, which is edge-disjoint to C. Let x and y be two neighbouring vertices of C, both of which are different from v. Denote: E = the edge $\{x,y\}$, X = xv - path in C, which does not contain y, and Y = yv path in C, which does not contain x. Then both of the feasible sets $P \cup X \cup E$ and $P \cup Y \cup E$ do exist in the UBG, while at most one of them does in a DBG.

Thus the family of feasible sets of a DBG is a proper subfamily of the family of feasible sets of the corresponding UBG and DBG and UBG cannot be isomorphic in the considered case. ∎

Consequence of Lemmas 5.2 and 5.3:

Our consideration of non-isomorphic orientations should be restricted exactly to those cases when there exists a cycle of a length greater than 2 (there are only 16 undirected graphs to be considered in our list of 54 undirected graphs).

The following lemma says that each undirected rooted graph can be "normally" oriented:

Lemma 5.4. *Given a pair (G, r), where G is a loopless connected undirected graph and r is the root, there always exists an orientation of G such that the directed branching greedoid associated to the pair (G, r) is normal.*

Proof. A generalized spanning tree is a subgraph T of G, which is incident to each vertex of G, the only cycles of which (if any) are of length 2, and which satisfies: if an edge x belongs to T, then all the edges of G, which are parallel to x, also belong to T.

We primarily orient the edges of a generalized spanning tree T - FROM the root r (towards outside). Each of the remaining undirected edges $\{x,y\}$ (appart from the edges of T) is oriented in the following manner:

IF x and y lie on the same directed path in T from the root r THEN
the edge $\{x,y\}$ should be oriented in the same direction as the xy-path within T

ELSE the edge $\{x,y\}$ may be oriented arbitrarily.

It is obvious that the $\{x,y\}$ oriented in this manner can be attached to the oriented path leading from the root to the source of $\{x,y\}$, without making an oriented cycle in the enlargened path. This implies that each of G belongs to a feasible set in the constructed DBG. #

Consequence of Lemmas 5.2, 5.3 and 5.4:

The number of non-isomorphic (normal, respectively general) DBG's corresponding to a basic unoriented graph G (with different choices of the root) - is not smaller than the corresponding number of UBG's. Generally speaking, the number of non-isomorphic DBG's on a ground-set is not smaller than the number of corresponding UBG's.

Our next lemma facilitates the consideration of non-isomorphic orientations by introducting some necessary conditions for the orientations leading to normal DBG's. We are able by applying it to fix the necessary orientation on certain edges:

Lemma 5.5. *All the orientations of the edges of a rooted undirected graph, which correspond to a normal directed greedoid, must obey the following rules:*

(1) Each edge incident to the root must be oriented FROM it (towards outside).

(2) Each generalized bridge (i.e., a usual bridge which might be replaced by a bundle of mutually parallel edges) must be oriented FROM the root.

(3) Each edge incident to a generalized bridge and following it (in the sense of the orientation of the bridge) must be oriented FROM the root.

(4) Each vertex must be reachable by an oriented path from the root.

Proof. The reachability requirement (4) is obvious, while (2) is a useful special case of it. If the greedoid is normal, then violating some of the rules (1) and (3) would mean that an arborescence contains an oriented cycle, a contradiction. #

Remark. The necessary conditions for "normal" orientations given in Lemma 5.5 are by no means sufficient. For example, consider the directed graph on the vertex-set $\{a,b,c,d\}$ with oriented edges (ab), (ad), (cb), (cd), (dc) and with the root a. The underlying undirected graph contains no bridges, the conditions (1) and (4) are satisfied. Nevertheless, the oriented edge (cd) is not included into a feasible set, that is, the DBG is not normal.

The only 16 non-isomorphic undirected (connected loopless) graphs on at most five edges, which have a cycle of a length greater than 2 - give rise to the total of 40 non-isomorphic rooted undirected graphs. The rules of Lemma 5.5. have not fixed the orientation of one, two, three edges in 24, 12, 4 of these 40 cases respectively.

Given a pair (G,r), where G is an undirected graph and r is the root, we primarily determined the fixed orientations of edges and later searched for all those non-isomorphic ways to orient the remaining edges, which leave the corresponding DBG normal. We shall give an example of such a search:

Let the undirected graph G on the vertex-set $\{a,b,c,d\}$ with the edges (a,b), (a,c), (a,d), (b,c), (b,d) (this graph is denoted as G 33 in our list) be given and let the vertex c be the root. The oriented edges (ca) and (cb) are compulsory for a "normal" orientation. The "flexible" orientations of the other three edges are treated as follows:

The vertices *a* and *b* are in isomorphic positions w.r.t. the pair (G,c) (more precisely, *a* and *b* belong to the same orbit of the subgroup of the automorphism group on the vertex-set of *G*, consisting of those automorpisms, which fix the root *c*). Therefore, we may fix an arbitrary of the oriented edges (ab) and (ba), say (ab). The other possibility should be neglected, since it leads to isomorphic cases.

There are four possibilities to orient the last two edges:

(ad) (bd) - (ad) (db) - (da) (bd) - (da) (db).

We throw away the third possibility, since it is isomorphic to the second one, as well as the fourth possibility, since it makes the vertex *d* unreachable from the root *c* (and the corresponding DBG non-normal). Thus in our list with the rooted undirected graph (*G* 33, *c*) we have the fixed oriented edges (ca) and (cb), while the collection of non-isomorphic flexible orientations is

(ab) (ad) (bd) - (ab) (ad) (db)

Finally, we shall give the list of non-isomorphic DBG's, which are not UBG's. The corresponding graphs by Lemma 5.2. have a cycle of a length greater than 2 and, by Lemma 5.3., all the undirected graphs on at most 5 edges, which have such a cycle, should be included. Each row of the list corresponds to those DBG's, which have the common underlying undirected graph (the denotations of which are the same as in the Section 4), the common root and the common fixed oriented edges (the orientation of each of these edges is in accordance with Lemma 5.5). Oriented edges (in brackets) are not separated by commas. Different DBG's in the same row are separated by short lines, which stand between their collections of flexibly oriented edges:

LIST OF DIRECTED BRANCHING GREEDOIDS

(which are not UBG's)

Underlying undirected graph	Root	Fixed oriented edges	Flexibly oriented edges (corresponding to the possible non-isomorphic orientations)
G 6	a	(ab) (ac)	(bc)
G 13	a	(ab) (ac) (ad)	(bc)
G 13	b	(bc) (ba) (ad)	(ac) - (ca)
G 13	d	(da) (ab) (ac)	(bc)
G 14	a	(ab) (ac)	(bd) (dc) - (bd) (cd)
G 21	a	(ab) (ab) (ac)	(bc) - (cb)
G 21	c	(ca) (cb)	(ab) (ab) - (ab) (ba)
G 28	a	(ab) (ac) (ad) (de)	(bc)
G 28	b	(bc) (ba) (ad) (de)	(ac) - (ca)
G 28	d	(da) (de) (ab) (ac)	(bc)
G 28	e	(ed) (da) (ab) (ac)	(bc)
G 29	a	(ab) (ac) (ad) (ae)	(bc)
G 29	b	(ba) (bc) (ad) (ae)	(ac) - (ca)
G 29	d	(da) (ab) (ac) (ae)	(bc)
G 30	a	(ab) (ac) (ad) (be)	(bc) - (cb)
G 30	c	(ca) (cb) (ad) (be)	(ab)
G 30	d	(da) (ab) (ac) (be)	(bc) - (cb)
G 31	a	(ab) (ac) (ad)	(be) (ce) - (be) (ec)
G 31	b	(be) (ba) (ad)	(ac) (ce) - (ac) (ec) - (ca) (ec)
G 31	d	(da) (ab) (ac)	(be) (ce) - (be) (ec)
G 31	e	(eb) (ec) (ad)	(ba) (ac) - (ba) (ca)
G 32	a	(ab) (ac)	(bd) (de) (ce) - (bd) (de) (ec)
G 33	a	(ab) (ac) (ad)	(cb) (bd) - (cb) (db) - (bc) (bd)
G 33	c	(ca) (cb)	(ab) (ad) (bd) - (ab) (ad) (db)
G 43	a	(ab) (ab) (ab) (ac)	(bc) - (cb)
G 43	c	(ca) (cb)	(ab) (ab) (ab) - (ab) (ab) (ba)
G 44	a	(ab) (ab) (ac) (ac)	(bc)
G 44	b	(ba) (ba) (bc)	(ac) (ac) - (ca) (ca) - (ac) (ca)
G 51	a	(ab) (ac) (ad)	(bc) (bc) - (bc) (cb)
G 51	b	(ba) (bc) (bc) (ad)	(ac) - (ca)
G 51	d	(da) (ab) (ac)	(bc) (bc) - (bc) (cb)
G 52	a	(ab) (ab) (ac) (ad)	(bc) - (cb)
G 52	b	(ba) (ba) (bc) (ad)	(ac)- (ca)
G 52	c	(ca) (cb) (ad)	(ab) (ab) - (ba) (ba) - (ab) (ba)
G 52	d	(da) (ab) (ab) (ac)	(bc) - (cb)
G 53	a	(ab) (ab) (ac) (ad)	(cd)
G 53	b	(ba) (ba) (ac) (ad)	(cd)
G 53	c	(ca) (cd) (ab) (ab)	(ad) - (da)
G 54	a	(ab) (ab) (ac)	(bd) (dc) - (bd) (cd) - (cd) (db)
G 54	c	(ca) (cd)	(ab) (ab) (bd) - (ab) (ab) (db) - (ba) (ba) (db) - (ab) (ba) (db)

6. Convex shellings

Let "*chull* (X)" be an abbrevation for "the convex hull of X". A pair (C,r) is a *circuit* with the root r ([7]) of a convex shelling on E if C is a minimal subset of E satisfying: r is the only element of C which is not a vertex of *chull* (C).

Lemma 6.1. *A convex shelling (E,F) is uniquely determined by the family of its (rooted) circuits (C,r), where $r \in C \subseteq E$, in the following manner:*

$X \subseteq E$ is NOT in F if and only if there exists a circuit (C,r) such that $X \cap C = \{r\}$.

Proof. IF-part: Let $C-r = \{c[1],\dots,c[k]\}$. Then the root r cannot become a vertex of *chull* $(E-\{x[1],\dots,x[i-1]\})$ before at least one $x[p]$ $(1<=p<=i-1)$ becomes equal to some $c[q]$ $(1<=q<=k)$ (otherwise it is impossible to approach the root r; it remains encircled with the elements of $C-r$, since

chull $(C) \subseteq$ *chull* $(E-\{x[1],\dots,x[i-1]\})$).

ONLY-IF-PART: We primarily make preparations by extracting a sequence S of vertex-set of convex hulls on the basis of the following algorithm:

```
Z := E;    i:= 1;
REPEAT
   Let S[i] be the set of vertices of chull (Z);
   Z := Z - S[i] ;
   i:= i+1
UNTIL Z = ø
```

Now assume that some $X \subset E$ satisfies for each circuit (C,r):

If $r \in X$, then $X \cap (C-r) <> \emptyset$

Then the ordering $x[1]\dots x[k]$ of X with the property:

$x[i]$ is the vertex of *chull* $(E-\{x[1],\dots,x[i-1]\})$, $1<=i<=k$,

can be constructed by the following algorithm:

```
Y := X;    i: = 1;
REPEAT
   T[i] := X ∩ S[i];
   Pick up the elements of T[i] in an arbitrary order;
   (that is, make an arbitrary ordering of T[i] be the next
   section of ordering of X that we construct)
   Y := Y - T[i] ;
   i: = i+1
UNTIL Y = ø.
```

We should confirm that our picking up of elements of T[i] is legal, i.e., that we never pick up the root of a circuit before some of the other elements of that circuit have already been picked up:

The elements of T[1] are not roots of circuits. Given an element x of T[i] (i>1), we know that x is a vertex of the i-th convex hull. This implies that each non-root element of $X \cap C$, for any circuit C with the root x -

lies in a set $T[j]$ for some j smaller than i. It follows by our algorithm that all the elements of $(X \cap C) - x$ have been picked up before x. The assumption of our ONLY-IF part says that $|(X \cap C) - x| >= 1$, that is, at least one outer element of C has been picked up before the root x. #

Remarks. We need not have all the circuits to describe a convex shelling; special, so-called, critical circuits are sufficient ([7]). For example, if we have the circuits (abc,a) and (bde,b), then the circuit (acde,a) is a consequence.

If all the 1-subsets of the ground-set E are feasible, then the convex shelling has no circuits and F becomes equal to the power set of E. This is a special case of the fact that all the interval greedoids with all the 1-sets feasible - are matroids. Namely, the only matroids which are shelling structures (in particular, convex shellings) at the same time - are "free" matroids, which have all the subsets of the ground-set feasible.

The following lemma considerably reduces our search:

Lemma 6.2. *The circuits of convex shellings have at least three elements.*

Proof. 1-circuits would contradict the given definition of a circuit, since r is the vertex of *chull* $(\{r\})$.

Let a 2-circuit $\{x,y\}$ with the root x be given. Then y is the only vertex of *chull* $(\{x,y\})$. It follows that x coincides with y, but then x should also be a vertex of *chull* $(\{x,y\})$, a contradiction with the definition of the root. □

When listing the non-isomorphic convex shellings on at most 5 elements, we shall concentrate on those of them which have the 5-element ground-set. If the union of all the circuits of a convex shelling CS is a k-set ($k < 5$), then CS can be bijected to a convex shelling on i elements, for each i, $k <= i <= 5$.

For the sake of shortness, we shall further replace the circuit denotation "(C,x)" by "x-$(C-x)$. E.g., we shall write "*a-bc*" instead of "(*abc*,*a*)".

We proceed with the details of our search:

There is one circuit-free convex shelling on 5 elements.

If there is exactly one circuit C, then the convex shelling is determined by the cardinality of C. There are three possibilities ($|C| \in \{3,4,5\}$).

If there are some two 3-circuits $C1$ and $C2$, then $|C1 \cap C2|$ belongs to the set $\{1,2\}$ (because of $|C1 \cup C2| <= 5$). Note that 3-circuits correspond to triples of collinear points.

If $|C1 \cap C2| = 2$, then we have two triples of collinear points, which have two points in common. It follows that all the four points are collinear. Such a 4-point configuration includes four 3-circuits. If $k=4$, then this is the unique remaining possibility.

If $|C1 \cap C2| = 1$, then assume that the points of $C1 \cup C2$ are not all collinear (the collinear case will be considered afterwards). We have $k=5$. The only common point for the (different) lines supporting the triples $C1$ and $C2$ may be:

a) the root for both $C1$ and $C2$

b) a vertex for both $C1$ and $C2$

c) the root for one triple and a vertex for the other.

Thus, there are three non-isomorphic possibilities. Note that another 4-circuit necessarily exists with the case c).

Now, assume that $k=5$ and that there exist at least three 3-circuits $C1, C2, C3$:

Suppose that $|C1 \cap C2| = 1$. Then $|C1 \cup C2| = 5$ and at least one of the cardinalities $|C1 \cap C3|$ and $|C2 \cap C3|$ is equal to 2. Let $|C1 \cap C3| = 2$. It follows that $|C1 \cup C3| = 4$ and that the four points of $C1 \cup C3$ are collinear on some line p. If the line q supporting $C2$ is different from p, then the lines p and q have at most one common point and we have that $k>=6$, a contradiction.

If each two of the circuits $C1, C2, C3$ have a 2-intersection, we derive by two applications of a former reasoning that all the five points are collinear.

We conclude that 5 collinear points (having 10 3-circuits) are the only possibility with $k = 5$ and at least three 3-circuits.

There remains to consider the cases when $k = 5$, there is a 4-circuit (or a 5-circuit), which is not the only circuit, and there is no more than one 3-circuit.

Let a 4-circuit x-abc be given and consider the position of the fifth point y. We may assume that the points a, b, c belong to the set V of vertices of *chull* $(abcxy)$. Namely, the existence of a 4-circuit gives $|V|>=3$. The root x cannot belong to V, since it is encircled by the triangle abc. If, for example, $V = aby$, then we also have the 4-circuit x-aby and we may interchange the denotations c and y. Thus, we have to consider the possibilities $V = abcy$ and $V = abc$.

(1). $V = abcy$. If y does not belong to the abc-plane, then $abcx$ is the only circuit. Otherwise, $abcy$ is a planar convex 4-gon and we may assume (since the denotations a, b, c are "flexible") that ac and yb are its (inner) diagonals. If x is incident to the line yb, then we have the 3-circuit x-by, otherwise we have exactly one of the 4-circuits *x-aby* and *x-bcy*. Thus we obtained two non-isomorphic cases with (I):

x-abc, *x-by* and *x-abc*, *x-aby* .

(II). *V* =*abc*. We primarily consider the subcase when *y* belongs to one of the edges of the triangle *abc*, say *ab*. Then we distinguish two possibilities: the point *y* either belongs to the line *cx* or not. The first of them gives two 3-circuits: *x-cy* and *y-ab*, and is isomorphic to the formerly considered possibility c) for two non-collinear 3-circuits. The second possibility gives exactly one of the 4-circuits *x-acy* and *x-bcy*. Thus the only convex shelling obtained here is

x-abc, *x-acy*, *y-ab* .

If *y* (as well as *x*) belongs to the interior of the triangle *abc*, then the line *xy* either contains a vertex (say *a*) or intersects (the interiors of) two edges (say *ac* and *bc*). In the first case we may assume that the point *y* lies between *a* and *x*. Then, we have *y-ax*, *y-abc*, *x-abc*, *x-bcy*. In the second case we may assume the existence of the 4-circuit *x-acy*. Then, we have *x-abc*, *y-abc*, *x-acy*, *y-bcx*. Thus case (II) gave three new non--isomorphic convex shellings.

Finally, the existence of a 5-circuit *x-abcd* implies that the vertices *a*,*b*,*c*,*d* belong to *chull* (*abcdx*) and none of them can be the root of a circuit (unless some sixth element exists). If follows that *x-abcd* is the only circuit.

The following list of non-isomorphic convex shellings (given by families of their circuits) summarizes the results of the considerations above:

1) ø
2) *a-bc*.
3) *a-bcd*.
4) *a-bcde*.
5) *a-bc*, *b-ad*, *a-cd*, *b-cd*.
6) *a-bc*, *a-bd*, *a-ce*, *a-de*, *b-ae*, *b-ce*, *b-de*, *c-ad*, *c-bd*, *c-de*.
7) *a-bc*, *a-de*.
8) *a-cd*, *b-ce*.
9) *a-bc*, *b-de*, *a-cde*.
10) *a-be*, *a-bcd*.
11) *a-bcd*, *a-bce*.
12) *b-cd*, *a-cde*.
13) *a-bc*, *a-cde*, *b-cde*, *b-ade*.
14) *a-bcd*, *b-ace*, *a-cde*, *b-cde*.

In addition to the considered classes of interval greedoids, we remark that the matroid enumeration is completed up to 8 elements (c.f. [1]), while the shelling structures are extracted as full interval greedoids and are enumerated (up to 5 elements) in [3].

We shall give the total number of non-isomorphic greedoids on *n* elements for 0<=*n*<=5 in all of the considered classes:

n	0	1	2	3	4	5
General greedoids ([2])	1	2	5	20	228	25612
Interval greedoids ([3])	1	2	5	18	132	4511
Local poset greedoids	1	2	5	17	90	906
Poset greedoids	1	2	4	9	25	88
Undirected branching gds.	1	2	5	14	44	148
Directed branching gds.	1	2	5	14	48	179
Convex shellings	1	1	1	2	4	14
Matroids ([1])	1	2	4	8	17	38
Shelling structures ([3])	1	1	2	6	34	672

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Rezime

JEDNA KLASIFIKACIJA INTERVALNIH GRIDOIDA NA SKUPOVIMA OD NAJVIŠE 5 ELEMENATA

U radu je data konstrukcija svih neizomorfnih gridoida na najviše 5 elemenata, koji spadaju u neku od sledećih pet klasa intervalnih gridoida:

lokalni poset gridoidi, orijentisani razgranati gridoidi i konveksne školjkaste strukture. Gridoidi u prve dve klase su određeni uz pomoć računara, na osnovu ranije generisanog kataloga [3] svih neizomorfnih intervalnih gridoida na skupovima od najviše 5 elemenata. Ostale tri klase gridoida su "ručno" konstruisane, na osnovu nekih opštijih teoretskih razmatranja (koja nisu ograničena na skupove od najviše 5 elemenata).

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