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ABELIAN THEOREMS FOR THE STIELTJES -HILBERT TRANSFORM OF DISTRIBUTIONS

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Abstract

Two theorems of the Abelian type are proved using the S-asymptotic behaviour of distributions. First, a definition is given of the Stieltjes-Hilbert transform which generalizes the classical Stieltjes and Hilbert transform. Then, a structural theorem for distributions having S-asymptotic is proved.

1. Introduction

In the last thirty years many definitions of the asymptotic behaviour of distributions have been presented. We can roughly divide them into two sets. Representatives of the first set are definitions given by M.J. Lighthill [7] and by J. Lavoine and O.P. Misra [6]. Representatives of the second set are the quasiasymptotic [14] and S-asymptotic [9]. The S-asymptotic is also called the shift or L. Schwartz asymptotic.

Alredy in his book [11, II,p.56] L. Schwartz used the notion we call the S-asymptotic. Yu.A. Brichkov and Yu.M.Shirokov [3] studied the S-asymptotic expansion of a distribution as a new approach to the quantum field theory.

It is also possible to define the Stieltjes and Hilbert transform of distributions in various ways. One of them is the so-called direct approach in which we construct a basic space $A \supset D$ of functions to which the set $\{(s+t)^{-r-1},\ Im\ s \neq 0,\ r \geq 0\}$ belongs. Then we define a transform for the

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generalized function T belonging to the dual space A' by the expression $\langle T(t), (s+t)^{-r-1}\rangle$. If the constructed space A' is such that $T \subset A'$ has the support belonging to $[0,\infty)$, we have the generalized Stieltjes transform (see [6], [8], [10], [16]). If supp $T \subset (-\infty,\infty)$ and $r=0$, this is a Hilbert transform and for $r \neq 0$, the generalized Hilbert transform (see [1], [2], [5]).

We shall define and use the Stieltjes-Hilbert transform which is a generalization of the classical Stieltjes and Hilbert transform, as well. This definition contains all of those, defined in the direct method, for which $\eta_\omega(s+t)^{-1}$ converges to $(s+t)^{-1}$ in A (for η_ω see page 3). In such a way, our Abelian type theorems are valued for all those transforms.

2. Notations and definitions

N is the set of natural numbers; R the set of real numbers and C the set of complex numbers. We shall denote by I one of the intervals $[0,\infty)$, $(-\infty,0]$, $(-\infty,\infty)$, and by P the set of all the real and positive functions defined on R. For different spaces of distributions, we shall use the usual notation (see for example [11]).

The following relations and properties will be used:

i) $\langle T(x+h), \varphi(x)\rangle = (T*\hat{\varphi}), \ h \in R; \ \hat{\varphi}(x) = \varphi(-x)$, $x \in R$, where $T \in D'$, $\varphi \in D$ and the asterisk denotes the sign for the convolution [11, II,p.22].

ii) If $T \in D'_{L^p}$, $\varphi \in D_{L^q}$, $1 \le p,q \le \infty$ and $\frac{1}{p} + \frac{1}{q} - 1 \ge 0$ then $(T*\varphi) \in D_{L^r}$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and $T\varphi \in D'_{L^s}$, $s \ge 1$, $\frac{1}{s} \le \frac{1}{p} + \frac{1}{q}$ [11], II,p.59. The space D'_{L^∞} is denoted by B'.

iii) If $\varphi \in D_{L^p}$, $1 \le p < \infty$, then $\varphi(x) \to 0$ when $|x| \to \infty$ [11, II,p.55].

Definition 1. *A distribution $T \in D'$ has the S-asymptotic in I relative to the function $c \in P$ and with the limit $U \in D'$, if the following limit exists*

$$(1) \qquad \lim_{h\in I, |h|\to\infty} \langle T(x+h)/c(h), \varphi(x)\rangle = \langle U,\varphi\rangle, \ \varphi \in D .$$

Then we write $T(x+h) \overset{s}{\sim} c(h)\cdot U(x)$, $h \in I$ and we say that T has the S-asymptotic in D' [9].

We shall use the same definition for a $T \in B'$ and $\varphi \in D_{L^1}$ stressing that T has the S-asymptotic in B'.

Suppose that $T \in D'$ and has the S-asymptotic with the limit $U \neq 0$, then, there exists a $\varphi \in D$ such that $\langle U,\varphi\rangle \neq 0$ and $\langle U(x+t),\ \varphi(x)\rangle \neq 0$ for t belonging to a neighbourhood $V(h_0)$ of $h_0 \in R$. For this φ, from relation:

$$\lim_{h\in I,\ |h|\to\infty} \frac{c(h+t)}{c(h)} \langle \frac{T(x+(h+t))}{c(h+t)},\ \varphi(x)\rangle =$$

$$= \lim_{h\in I,\ |h|\to\infty} \langle \frac{T((x+t)+h)}{c(h)},\ \varphi(x)\rangle,\quad t \in V(h_0),$$

it follows that

$$\lim_{h\in I,\ |h|\to\infty} c(h+t)/c(h) = d(t),\quad t \in V(h_0)$$

exists and that U satisfies the equation:

(2) $$d(t)\langle U,\phi\rangle = \langle U(x+t),\ \varphi(x)\rangle,\quad t \in V(h_0)$$

U, as a distribution, has all the derivatives. From relation (2) it follows that

(3) $$d(t) = exp(at) \quad \text{and} \quad U(t) = C\ exp(at),\quad t \in R,$$

where a is an element from R and C is a constant [8].

3. Stieltjes-Hilbert transform of distributions

In the following we shall use the well known function $\eta_\omega \in C^\infty$, $\omega > 0$ [15]:

$$\eta_\omega(x) = \int_{-2\omega}^{2\omega} q_\omega(x-t)dt,\quad x \in R,$$

where

$$q_\omega(x) = \begin{cases} D\omega^{-1} exp\left(-\dfrac{\omega^2}{\omega^2 - x^2}\right), & |x| < \omega \\ 0 & |x| \geq \omega \end{cases}$$

and $D\int_R q_1(t)dt = 1$.

The function η_ω has the properties: $0 \leq \eta_\omega(x) \leq 1$, $x \in R$; $\eta_\omega(x) = 1$, $x \in (-\omega,\omega)$; $\eta_\omega(x) = 0$, $|x| \geq 3\omega$; $|D^k\eta_\omega(x)| \leq C_k\omega^{-k}$, $x \in R$. The constants C_k do not depend on ω.

Definition 2. *The Stieltjes-Hilbert transform of a distribution $T \in D'$ (S_ρ-trasform) is defined by the limit:*

$$(4)\qquad \lim_{\omega\to\infty} \langle T(x),\ \eta_\omega(x)(s+x)^{-(\rho+1)}\rangle = S_\rho(T)(s),\ s\in \mathbb{C}\setminus R,$$

if it exists for a $\rho\in R$.

Remarks. a) s can belong to a larger set, as well. This depends on the support of T. So $S_\rho(\delta)(s) = s^{-(\rho+1)}$, $s\neq 0$.

b) If T is defined by the function f, $\operatorname{supp} f\subset [0,\infty)$, Definition 2 gives the classical Stieltjes transform, if it exists. Let $s\in \mathbb{C}\setminus(-\infty,0]$, then

$$S_\rho(f)(s) = \lim_{\omega\to\infty}\int_0^{3\omega} f(t)\eta_\omega(t)(s+t)^{-(\rho+1)}dt =$$

$$= \lim_{\omega\to\infty}\int_0^{\omega} f(t)(s+t)^{-(\rho+1)}dt +$$

$$+ \lim_{\omega\to\infty}\int_\omega^{3\omega} f(t)\eta_\omega(t)(s+t)^{-(\rho+1)}dt .$$

We have only to prove that:

$$(5)\qquad \lim_{\omega\to\infty}\int_\omega^{3\omega} f(t)\eta_\omega(t)(s+t)^{-(\rho+1)}dt = 0,\ s\in \mathbb{C}\setminus(-\infty,0]$$

when the classical Stieltjes transform exists.

Since for $\omega\le x\le 3\omega$

$$\eta_\omega(x) = \int_{-2\omega}^{2\omega} q_\omega(x-t)dt = \int_{(x-2\omega)/\omega}^{1} q_1(y)dy ,$$

the function $\eta_\omega(x)$ is positive and monotone decreasing in this interval. By the mean value theorem, there exists a ξ, $0<\xi<2$ such that

$$\int_\omega^{3\omega} \eta_\omega(t)\ \mathrm{Re}\Big[f(t)(s+t)^{-(\rho+1)}\Big]dt = \eta_\omega(\omega)\int_\omega^{\omega+\xi\omega} \mathrm{Re}\Big[f(t)(s+t)^{-(\rho+1)}\Big]dt .$$

The last integral tends to zero when $\omega\to\infty$, because we supposed that the Stieltjes transform of the function f exists. We have the same situation with the imaginary part of the integral from relation (5).

c) J. Lavoine and O.P. Misra [6] defined the Stieltjes transform of distributions belonging to a subset J' of D' which is used in many papers. A distribution T belongs to J' if and only if $\operatorname{supp} T\subset [0,\infty)$, $T = D^m G$, where G is a function having a support in $[0,\infty)$ and $G(x)(x+b)^{-r-m-1} \in L^1(R_+)$, $b > 0$. The Stieltjes transform of $T \in J'$ is:

$$(6) \qquad S_r(T) = (r+1)\dots(r+m)\int_0^\infty G(t)(s+t)^{-r-m-1}dt, \; s \in C\setminus(-\infty,0].$$

Using properties of the function η_ω, it is easy to see that (4) gives precisely (6). A similar situation holds with the definitions of the Stieltjes transform in [8], [16] and for the Hilbert transform in [1], [2].

4. Distribution having the S-asimptotic

We shall prove, first, a structural theorem for the distributions having the S-asymptotic.

Theorem 1. *Suppose* $T_0 \in D'$. *If* $T_0(x+h) \overset{s}{\sim} c(h)\cdot 0$, $h \in R$ *for any* $c \in P$ *such that* $c(h) \to \infty$, $|h| \to \infty$ *and* $T_0(x+h) \overset{s}{\sim} 1\cdot C$, $h \in I$, *then:*

a) $T_0 \in B'$ *;*

b) $T_0 = \sum_{i=0}^{2} D^{ik} F_i$, *where* F_i *are continuous functions belonging to* L^∞;

c) For every $0 \le i \le 2$, $F_i(x+h)$ *converges uniformly to a constant when* x *belongs to any interval* $[-r,r]$ *and* $h \in I$, $|h| \to \infty$.

d) T_0 *has the S-asimptotic in* B', *as well, related to* $c(h) = 1$ *and with the limit* $U = C$, *in the interval* I.

Proof. a) By relation 1) in 2 $\langle T_0(x+h),\varphi(x)\rangle = (T_0*\hat\varphi)(h)$ for every $\varphi \in D$ and $h \in R$. First, we shall prove that $(T_0*\hat\varphi)(h) \in L^\infty$ for every $\varphi \in D$. Suppose, on the contrary, that $(T_0*\hat\varphi)(h)$ is not a bounded function. Then, there exist two sequences $\{h_m\} \subset R$, $|h_m| \ge m$ and $\{c_m\} \subset R$, $c_m \to \infty$, when $m \to \infty$ such that $(T_0*\hat\varphi)(h_m) = c_m$. Now, for the $c_0(h)$, $c_0(h_m) = \sqrt{|c_m|}$ the limit

$$\lim_{|h|\to\infty} \langle T_0(x+t)/c(h),\ \varphi(x)\rangle$$

does not exist. But, by our supposition, this limit has to exist and to be zero.

From $(T_0*\hat\varphi) \in L^\infty$ it follows that $T_0 \in B'$ (see Theorem XXV form [11, II, p.57] and the set of distributions $Q = \{T_h = T_0(x+h),\ h \in R\}$ is weakly bounded and bounded in D'.

b) In addition to Q we shall construct an other bounded set of distributions. We denote by $S = \{\psi \in D,\ \|\psi\|_{L^1} \le 1\}$. We have seen that for a fixed $\varphi \in D$, $(T_0 * \hat{\varphi}) \in L^\infty$. Now, for every $\psi \in S$:

$$|\langle T_0 * \hat{\psi}, \varphi\rangle| = |\langle T_0 * \hat{\varphi}, \psi\rangle| = \left|\int_R (T_0 * \hat{\varphi})(t)\psi(t)dt\right| \le \|T_0 * \hat{\varphi}\|_{L^\infty} \|\psi\|_{L^1}.$$

Hence, the set of regular distributions, defined by the set of continuous functions $H = \{U_\psi = T_0 * \hat{\psi},\ \psi \in S\}$ is weakly bounded and bounded in D'.

A set $W \in D'$ is bounded if and only if for every $\alpha \in D$ the set of functions $\{T*\alpha\ ,\ T \in W\}$ is bounded on every compact set M belonging to R [11, II, p.50]. Hence, $\{T*\alpha\ ,\ T \in W\}$ defines a bounded set of regular distributions. In such a way $\{T_h * \varphi,\ T_h \in Q\}$ and $\{U_\psi * \varphi,\ U_\psi \in H\}$ give two bounded sets of regular distributions. Now, for these two sets we can repeat twice a part of the proof of Theorem XXII from [11, II, p.51].

We denote by Ω an open neighbourhood of zero in R which is relatively compact in R, $cl\Omega = K$, K a compact set. Then, by the mentioned part of the proof in [11] there exists $m_1 \ge 0$ and $m_2 \ge 0$, such that the mapping $(\alpha, \beta) \to U_\psi * (\alpha * \beta)$ or $(\alpha, \beta) \to T_h * (\alpha * \beta)$ are equicontinuous and map $D_\Omega^{m_1} \times D_\Omega^{m_1}$ or $D_\Omega^{m_2} \times D_\Omega^{m_2}$ into L_E^∞; E is the interval $[-r, r]$, r a positive number. Hence, for every $x \in E$ and $h \in R$ the function $(T_h * \alpha * \beta)(x) = (T*\alpha*\beta)(x+h)$ is continuous.

Let $Z(0,\rho)$ be a ball in L_E^∞, then there exists a neighbourhood $V_1(m_1, \varepsilon_1, K_1)$ in $D_\Omega^{m_1}$, such that $U_\psi * (\alpha * \beta) \in Z(0,\rho)$ for $\alpha, \beta \in V_1(m_1, \varepsilon_1, K_1)$, $U_\psi \in H$ and a neighbourhood $V_2(m_2, \varepsilon_2, K_2) \subset D_\Omega^{m_2}$, such that $T_h * (\alpha * \beta) \in Z(0,\rho)$ for $\alpha, \beta \in V_2(m_2, \varepsilon_2, K_2)$, $T_h \in Q$. Let $K_0 = K_1 \cap K_2$, $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$ and $m = \max(m_1, m_2)$. We shall now use relation (VI, 6; 23) from [11, II]

$$T = \Delta^{2k} * (\gamma E * \gamma E * T) - 2\Delta^k * (\gamma E * \xi * T) + (\xi * \xi * T),$$

where E is a solution of the iterated Laplace equation: $\Delta^k E = \delta$; $\gamma, \xi \in D_\Omega$, suppγ and supp ξ belonging to $K_0 = K_1 \cap K_2$. We have only to choose the number k large enough so that $\gamma E \in D_\Omega^m$. Now, we can take: $F_2 = \gamma E * \gamma E * T_0$; $F_1 = \gamma E * \xi * T_0$ and $F_0 = \xi * \xi * T_0$. All of these fuctions are of the form: $F_i = T_0 * \alpha_i * \beta_i$; $\alpha_i, \beta_i \in V(m, \varepsilon_0', K_0)$, $\varepsilon_0' > 0$.

We have to prove that F_i have the properties given in Theorem 1. For $\alpha_i, \beta_i \subset V(m, \varepsilon_0', K_0)$ and $\psi \in S$

$$|\langle T_0 * \alpha_i * \beta_i), \psi\rangle| = |[(T_0 * \hat{\psi}) * (\hat{\alpha}_i * \hat{\beta}_i)]\ (0)| \le \rho(\varepsilon_0'/\varepsilon_0)^2 = M$$

Now, let $\mu\neq0$ be any element from L^1, then $\mu / \|\mu\|_{L^1} \in S$ and $|\langle(T_0*(\alpha_i*\beta_i)),\mu\rangle| \leq M\|\mu\|_{L^1}$ which proves that $T_0*(\alpha_i*\beta_i)$ belong to L^∞. Since $F_i = T_0*(\alpha_i*\beta_i)$, $\alpha_i, \beta_i \in V(m,c_0',K_0)$, F_i are continuous and belong to L^∞.

c) We shall continue investigating the properties of F_i. By the properties of the convolution we have that $F_i(x+h)= F_i*\tau_{-h}= T_0*(\alpha_i*\beta_i)*\tau_{-h}= =T_h*(\alpha_i*\beta_i)$ for $\alpha_i, \beta_i \in D^m_\Omega$; where τ_{-h} is the translation operator.

We have proved that the mappings $(\alpha, \beta) \to T_h*\alpha*\beta$, $T_h \in Q$, are equicontinuous and map $D^m_\Omega \times D^m_\Omega$ into L^∞_E. D is a dense subset of D^m, $m \geq 0$. We can construct a subset of D_K, $clQ=K$, which is dense in D^m_Ω. Since $T_h*(\varphi*\psi)\to C*\varphi*\psi$ for $\varphi*\psi \in D_\Omega \times D_\Omega$, then $T_h*\alpha_i*\beta_i$ converges to $C*\alpha_i*\beta_i$, as well (see [11], II p.53).

d) There remains to prove the last part of Theorem 1. For $\mu \in D_{L^1}$ and $T_0\in B'$, noting that $F_i\in L^\infty$, we have:

$$|\langle T_0(x+h),\ \mu(x)\rangle \leq \sum_{i=0}^{2} \int_R |F_i(x+h)\mu^{(i)}(x)|dx$$

$$\leq \sum_{i=0}^{2} M_i \int_R |\mu^{(i)}(x)dx\ ,$$

where $M_i= \sup |F_i(x)|$, $x \in R$. Hence, the set $\{ T(x+h),\ h \in R\}$ is weakly bounded in B'. Since D is dense in D_{L^1}, by the Banach-Steinhaus theorem the limit

$$\lim_{h\in I,\ |h|\to\infty} \langle T(x+h),\mu(x)\rangle,\ \ \mu \in D_{L^1}$$

exists, as well, and equals C.

Theorem 2. *For a $T \in D'$ we suppose:*

i) $T(x+h) \overset{\circ}{\sim} c(h)\cdot U(x),\ \ h \in I,\ \ U\neq0;$

ii) $T(x+h) \overset{\circ}{\sim} c_1(h)c(h)\cdot 0,\ \ h \in R,$ *where* $c_1(h)$ *is any function from* P, *such that* $c_1(h) \to \infty$, $|h| \to \infty$;

iii) For a $s_0\in C\setminus R$, $c(h)/(s_0+x+h)^r$ *is bounded for* $h \in R$ *and converges to* $C_1\neq0$ *when* $h \in I$, $|h| \to \infty$ *and* x *belongs to a compact set.*

Then, $T \in S'$ *and* $T(x)/(s_0+x)^r$ *is a distribution belonging to* B'. $T(x+h)/(s_0+x+h)^r \overset{\circ}{\sim} 1\cdot CC_1$, $h \in I$ *in* D' *and in* B', *as well.*

Proof. From supposition iii) of Theorem 2 it follows that the limit distribution $U=C$. Namely, for an $h^0 \in R$ and when x belongs to a compact set, $s_0 \in C \setminus R$:

$$\frac{c(h+h^0)}{c(h)} = \frac{c(h+h^0)}{(s_0+x+h+h^0)^r} \frac{(s_0+x+h)^r}{c(h)} \frac{(s_0+x+h+h^0)^r}{(s_0+x+h)^r} .$$

Hence, $\lim_{h\in I, |h|\to\infty} c(h+h^0)/c(h) = 1$. Relation (3) gives $U=C$.

By Theorem X from [11], I, p.72 (see also [9]), for a $\varphi \in D$, it follows the existence of the limit:

$$\lim_{h\in I, |h|\to\infty} \langle T(x+h)/(s_0+x+h)^r, \varphi(x)\rangle =$$

(7)

$$= \lim_{h\in I, |h|\to\infty} \langle \frac{T(x+h)}{c(h)}, \frac{c(h)}{(s_0+x+h)^r} \varphi(x)\rangle$$

Similarly, we show that

$$\lim_{|h|\to\infty} \langle T(x+h) \Big/ \left[c_1(h)\ (s_0+x+h)^r\right], \varphi(x)\rangle = 0, \quad \varphi \in D .$$

By Theorem XXV from [11], II, p.57 it follows that $T(x)/(s_0+x)^r \in B'$ (see also [12]) and by Theorem VI from from [11], II, p.95 it follows that $T \in S'$.

In such a way we prove that the distribution $T(x)/(s_0+x)^r$, $s_0 \in C\setminus R$, has the same properties as the distribution T_0 from Theorem 1 and we can use the assertion of this theorem which says that $T(x)/(s_0+x)^r$, $s_0 \in C\setminus R$, has the S-asymptotic in B' related to $c(h)$ and with the limit $U=CC_1$.

5. Abelian theorems for the Stieltjes-Hilbert transform

Theorem 3. *Suppose* $T \in D'$ *and*

i) $T(x+h) \overset{s}{\sim} c(h)\cdot U(x)$, $h \in I$, $U\neq 0$;

ii) $T(x+h) \overset{s}{\sim} c_1(h)c(h)\cdot 0$, $h \in R$, *where* c_1 *is any function from* P, *such that* $c_1(h) \to \infty$, $|h| \to \infty$;

iii) For a $s_0 \in C\setminus R$ *and* $r \geq 0$ $c(h)/(s_0+x+h)^r$ *is bounded for* $h \in R$ *and converges to* $C_1\neq 0$ *when* $h \in I$, $|h| \to \infty$ *and* x *belongs to any compact set in* R.

Then $T \in S'$ *has the* S_ρ*-transform for all* $\rho > r$, $S_\rho(T)(s)=\langle T(x)/(s_0+x)^r,$

$(s_0+x)^r/(s+x)^{\rho+1}\rangle$ and

$$\lim_{h\in I,\,|h|\to\infty} S_\rho(T)(s-h)/c(h) = 0, \quad \rho > r .$$

Proof. Since $|D^k\eta_\omega(x)| \le C_k\omega^{-k}$, $k > 0$, where C_k do not depend on ω, the set of functions $\eta_\omega(x)(s+x)^{r-\rho-1}$ converges to $(s+x)^{r-\rho-1}$, $\rho > r$, in D_{L^1}, when $\omega \to \infty$. Moreover, $(s_0+x)^r/(s+x)^r$, for $s_0, s \in C\setminus R$, belongs to D_{L^∞}; consequently

$$S_\rho(T)(s) = \lim_{\omega\to\infty} \langle T(x), \eta_\omega(x)(s+x)^{-\rho-1}\rangle$$

$$= \lim_{\omega\to\infty} \langle \frac{T(x)}{(s_0+x)^r}\frac{(s_0+x)^r}{(s+x)^r}, \eta_\omega(x)(s+x)^{r-\rho-1}\rangle =$$

$$= \langle T(x)/(s_0+x)^r, (s_0+x)^r(s+x)^{-\rho-1}\rangle.$$

Now, by Theorem 2 our distribution $T(x)/(s_0+x)^r$ fulfils the conditions for T_0 in Theorem 1 and:

$$\frac{1}{c(h)} S_\rho(T)(s-h) = \frac{1}{c(h)} \langle \frac{T(x+h)}{(s_0+x+h)^r}, (s_0+x+h)^r(s+x)^{-\rho-1}\rangle$$

(8)

$$= \sum_{i=0}^{2} \frac{(-1)^{ik}}{c(h)} \int_R F_i(x+h) D^{ik} \frac{(s_0+x+h)^r}{(s+x)^{\rho+1}} dx .$$

The expression $D^j\left[(s_0+x+h)^r(s+x)^{-\rho-1}\right]$ is given by the sum of elements which have the following form:

$$H_{j,p} = C_{j,p}(s_0+x+h)^{r-j+p}(s+x)^{-\rho-1-p}, \quad j \ge p \ge 0, \; j=0,k,2k.$$

We shall analyse $H_{j,p}/c(h)$ when $x \in R$ and $h \in I$

$$H_{j,p}/c(h) = C_{j,p} \frac{(s_0+x+h)^r}{c(h)} (s_0+x+h)^{p-j}(s+x)^{-\rho-1-p}$$

$$\le |C_{j,p}| \, |Im s_0|^{p-j} \frac{(|s_0+h|+1)^r}{c(h)} \frac{(|x|+1)^r}{|s+x|^{\rho+p+1}} .$$

This inequality shows that $H_{j,p}/c(h)$ is bounded by a function which belongs to L^1, when $h \in I$. Since $F_i(x+h)$ are bounded, as well, when $x \in R$, and $h \in I$, we can use the Lebesgue theorem for integral (8) to obtain that $S_\rho(T)(s-h)/c(h)$ tends to zero when $h \in I$, $|h| \to \infty$.

If we compare the results of Theorem 3 with the known results on Abelian type theorems at infinity by other authors (see for instance [10] in which these results are listed), we can establish that knowing the S-asymptotic of a distribution T on I having the S_ρ-transform, we know how $S_\rho(T)(s-h)$ behaves when $|h|\to\infty$, $h\in I$ for $s-h$ belonging to a set $\{|Ims|\ge\delta,\ \delta>0,\ h\in R\}$. With the quasiasymptotic we know the behaviour of the Stieltjes transform $S_\rho(T)(s)$ at most in the closed domain $-\pi+c\le args\le\pi-c$, $c>0$.

The next theorem presents more precisely how $S_\rho(T)(s-h)$ tends to zero when $|h|\to\infty$, $h\in I$.

Theorem 4. *Let $T\in D'$ and $s_0\in C\setminus R$. We suppose:*

i) $(s_0+x+h)\ T(x+h)\ \overset{s}{\sim}\ 1\cdot C,\quad h\in I$;

ii) $(s_0+x+h)\ T(x+h)\ \overset{s}{\sim}\ c_1(h)\cdot 0,\ h\in R$ *for any* $c_1\in P$, *such that* $c_1(h)\to\infty$, $|h|\to\infty$.

Then, $(s_0+x)T(x)\in B'$ *and* $S_\rho(T)(s-h)=o(h^{-1})$, $h\in I$, $|h|\to\infty$, $\rho>0$.

Proof. First, we shall show that $T\in D'_{L^v}$, $v>1$. By Theorem 1, $(s_0+x)T(x)\in B'$. Since $(s_0+x)^{-1}\in D_{L^q}$, for every $q>1$, it follows from property ii) in 2 that $(s_0+x)\ T(x)\ (s_0+x)^{-1}\in D'_{L^v}$, $v>1$.

The second step is to prove that

$$\langle T(x+h),(s+x)^{-\rho}\rangle\to 0,\ h\in I,\ |h|\to\infty. \tag{9}$$

For any $p>1/\rho$ and $\frac{1}{p}+\frac{1}{v}-1\ge 0$

$$\langle T(x+h),(s+x)^{-\rho}\rangle=(T*(s-x)^{-\rho})\ (h)\in D_{L^u},\ \frac{1}{u}=\frac{1}{v}+\frac{1}{p}-1. \tag{10}$$

Now, we have to prove that we can find p and v, such that $u<\infty$. Then, from iii) in 2 follows (9). For this we shall analyse two cases.

Case $\rho\ge 1$. In relation (10) we can take $p>1$, such that $1\le u<\infty$.

Case $0<\rho<1$. Since $\frac{1}{v}+\frac{1}{p}-1\ge 0$, we have $1>\frac{1}{v}\ge 1-\frac{1}{p}>1-\rho$, or $1<v<(1-\rho)^{-1}$. For such an v the number $\frac{1}{u}$ is strictlty positive, hence $1\le u<\infty$.

Now, we can write:

$$S_\rho((s_0+x)\ T(x))\ (s-h)=\langle(s_0+x+h)\ T(x+h),(s+x)^{-\rho-1}\rangle$$
$$=\langle T(x+h),(s+x)^{-\rho}\rangle+(s_0-s+h)\langle T(x+h),(s+x)^{-\rho-1}\rangle$$

Hence,

$$S_\rho(T)(s-h) = (s_0-s+h)^{-1}\ [S_\rho((s_0+x)\ T(x))\ (s-h)\ -\ \langle T(x+h),(s+x)^{-\rho}\rangle].$$

There remains only to prove that $S_\rho((s_0+x)T(x))(s-h) \to 0$, when $h \in I$, $|h| \to \infty$. By Theorem 1 d)

$$\lim_{h\in I,\,|h|\to\infty} S_\rho((s_0+x)T(x))(s-h) = \lim_{h\in I,\,|h|\to\infty} \langle (s_0+x+h)T(x+h),(s+x)^{-\rho-1}\rangle =$$

$$= \langle C,(s+x)^{-\rho-1}\rangle = 0.$$

The next example shows that Theorem 4 cannot be proved for $\rho = 0$

$$\int_0^\infty \frac{dx}{(x+a)(s+x)} = (a-s)^{-1} \ln\frac{a}{s}\ ,\ \ a,s > 0.$$

There arises an other question: if we know that $(s_0+x)^r T(x) \in B'$ for a $r > 1$, is it true that $S_\rho(T)(s-h) = o(h^{-r})$, $h \in I$, $|h| \to \infty$? The answer is negative. This shows the following integral:

$$\int_0^\infty e^{-t}\ \frac{dt}{(s+t)^\nu} = s^{-\nu/2}\ exp(s/2)\ W_{-\nu/2,(1-\nu)/2}(s)\ -s^{-\nu},\ \ s > 0,\ s \to \infty,$$

where $W_{\nu,\mu}$ is the Whittaker function.

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Rezime

ABELOVE TEOREME ZA STIELTJES-HILBERTOVU TRANSFORMACIJU DISTRIBUCIJA

Definiše se Stieltjes-Hilbertova transformacija distribucija:

Definicija 1. Stieltjes-Hilbertova transformacija distribucije T (S_ρ-transformacija) definiše se granicom

$$(1)\quad \lim_{\omega\to\infty} \langle T(x),\ \eta_\omega(x)(s+x)^{-(\rho+1)}\rangle = S_\rho(T)(s),\ s \in C\setminus R$$

ako postoji za $\rho \in R$. η_ω je poznata funkcija (vidi [15]).

U ovoj definiciji s može pripadati širem skupu, sve zavisi od nosača distribucije T. Ako je pak T regularna distribucija definisana funkcijom f i f ima klasičnu Stieltjesovu transformaciju, postoji i $S_\rho(f)$ i jednaka je sa klasičnom Stieltjes-ovom transformacijom.

Cilj rada je da se dokažu Abelove teoreme za ovako definisanu S_ρ-transformaciju koristeći se S-asimptotikom distribucija (vidi [9]). Prethodno je pokazana strukturna teorema za distribucije koje imaju S-asimptotiku.

Glavna teorema Abelovog tipa koja je dokazana je sledeća:

Teorema 1. Neka su zadovoljene sledeće pretpostavke o distribuciji T:

i) $T(x+h) \overset{s}{\sim} c(h)\cdot U(x),\ h \in I,\ U\neq 0$;

ii) $T(x+h) \overset{s}{\sim} c_1(h)c(h)\cdot 0,\ h \in R$, gde je c_1 pozitivna funkcija takva da $c_1(h) \to \infty,\ |h| \to \infty$;

iii) Za $s_0 \in (C\setminus R)$ i $r \geq 0$ $c(h)/(s_0+x+h)$ je ograničeno nad R i konvergira ka $C_1 \neq 0$, kada $h \in I$ i $|h| \to \infty$, a x pripada kompaktnom skupu u R.

Tada $T \in S'$ ima S_ρ-transformaciju za sve $\rho > r$, $S_\rho(T)(s) = \langle T(x)/(s_0+x)^r, (s_0+x)^r/(s+x)^{\rho+1} \rangle$ i

$$\lim_{h\in I,\ |h|\to\infty} S(T)(s-h)/c(h) = 0,\ \rho > r.$$

I je jedan od intervala $(-\infty,\infty)$, $(-\infty,a)$, (b,∞), $T(x+h) \overset{s}{\sim} c(h)\cdot U(x)$ označava da distribucija T ima S-asimptotiku u odnosu na $c(h)$ i sa granicom U.

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