Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 18,2, 197–203 (1988)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

REMARKS ON THE DISTRIBUTIONAL STIELTJES TRANSFORMATION AND THE S-ASYMPTOTIC

Stevan Pilipović
University of Novi Sad, Faculty of Science,
Institute of Mathematics, Dr I.Djuričića 4,
21000 Novi Sad, Yugoslavia

ABSTRACT

We investigate the behaviour of the Stiltjes transformation $S_r f$ of an $f \in S'_+$ whish has the S-asymptotic behaviour at ∞. As well we show that the domain of the convergence of $(S_r f)(z)$, $|z| \to \infty$, which is, by appropriate assumptions, of the form $\{re^{i\phi};\ r>0,\ |\phi|<\pi-\varepsilon\}$, $0<\varepsilon<\frac{\pi}{2}$, could not be enlarged to contain a half line $x+iy$, $y\neq 0$, $x\in(-\infty,0)$.

1. NOTIONS

Following [1] we define the space $J'(r)$, $r \in \mathbb{R}\setminus(-\mathbb{N})$, as a subspace of the space of tempered distributions with supports in $[0,\infty)$ (denoted by S'_+) consistend of all f of the form

(1) $\quad f = D^m F$ (D is the distributional derivative)

where

$F \in L^1_{loc}$, $\operatorname{supp} F \subset [0,\infty)$ and

---

AMS Mathematics Subject Classification (1980): 46F12, 44A15
Key words and phrases: Distributions, asymptotic behaviour, Stieltjes transformation.

(2) $\int_0^\infty |F(t)/(z+t)^{r+m+1}|dt<\infty, \quad z\in \mathbb{C}\setminus(-\infty,0]. \ (\mathbb{C}=\mathbb{R}+\mathbb{R}).$

The distributional Stieltjes transformation of an $f\in J'(r)$ (for which (1) and (2) hold) is a holomorphic function defined on $\mathbb{C}\setminus(-\infty,0]$ by

$$(S_r f)(z) = (r+1)_m \int_0^\infty (F(t)/(z+t)^{r+m+1})dt$$

where $(r)_m=r(r+1)\dots(r+m-1)$, $m\in\mathbb{N}$, and $(r)_0=1$.

REMARK. We can define $J'(r)$, $r\in\mathbb{R}\setminus(-\mathbb{N})$, as a subspace of S' consisted of all f for which (1) holds but without the assumption on $\operatorname{supp} F$ and for which (2) holds with $\operatorname{Im} z\neq 0$. In this case (3) defines the Stieltjes transformation of f which is a holomorphic function in $\mathbb{C}\setminus\mathbb{R}$ ([3]).

We studied in [6], [7], [2], [3], Abelian and Tauberian type results for this transformation by using the notion of the quasiasymptotic behaviour of a distribution from S'_+ and S' (see [10] and references there, and [5]). For the sake of simplicity we shall deal in this paper only with elements from S'_+. Recall that an $f\in S'_+$ has the quasiasymptotic at ∞ related to $k^\alpha L(k)$, $\alpha\in\mathbb{R}$, L is a slowly varying function (see [10]), if for some $g\in S'$, $g\neq 0$,

(4) $\lim_{k\to\infty} \langle f(kt)/(k^\alpha L(k)),\phi(t)\rangle = \langle g,\phi\rangle, \ \forall\phi\in S.$

Let $f\in S'_+$ have the quasiasymptotic at ∞ related to $k^\alpha L(k)$ and let $r\in\mathbb{R}\setminus(-\mathbb{N})$ such that $r>\alpha$. These assumption imply that $f\in J'(r)$ ([8]).

The following assertion is proved in[8] (with suitable $C\neq 0$):

$$(S_r f)(ks)/(k^{\alpha-r}L(k)) \to C(\Gamma(r-\alpha)/\Gamma(r+1))s^{\alpha-r}, \ k\to\infty \quad (C\neq 0).$$

(*) Moreover, if $L=1$ then

$(S_r f)(s)/s^{\alpha-r} \to C\Gamma(r-\alpha)/\Gamma(r+1), |s|\to\infty$, uniformly in any angle of the form $\Lambda_\varepsilon=\{\rho\exp(i\phi);\ \rho>0,\ -\pi+\varepsilon<\phi<\pi-\varepsilon\}$, $0<\varepsilon<\pi/2$.

We shall investigate in this note the distributional Stieltjes transformation of an f which has the S-asymptotic behaviour ([9]).

We shall say that an $f \in \mathcal{D}'$ ($f \in \mathcal{S}'$) has the S-asymptotic at ∞ in $\mathcal{D}'$ (in $\mathcal{S}'$) related to a positive continuous function $c(h)$, $h>0$, if for some $g \in \mathcal{D}'$ ($g \in \mathcal{S}'$), $g \neq 0$

$$\lim_{h\to\infty} \langle f(x+h)/c(h), \phi(x)\rangle = \langle g,\phi\rangle, \ \forall\phi \in \mathcal{D} \quad (\forall\phi \in \mathcal{S}). \tag{5}$$

(The S-asymptotic at $-\infty$ is defined in an adequate way.) We studied this notion in [9] and [6]. From now on we shall observe the S-asymptotic related to $c(h)$ in $\mathcal{S}'$. Note that in this case (5) implies the g=const. ([6]).

2. PROPOSITION 1. *Let* $f \in \mathcal{S}'_+$ *and have the S-asymptotic at* ∞ *in* $\mathcal{S}'$ *related to* $c(h)$ $(h>0)$. *Then there is* $r_0 \in \mathbb{R}$ *such that for any* $r>r_0$, $r \in \mathbb{R}\setminus(-\mathbb{N})$:

(i) $\lim_{h\to\infty} (S_r f)(z-h)/c(h) = 0, \quad z \in \mathbb{C}\setminus\mathbb{R};$

(ii) *If for some* C *and* α

$$(1+h^{\alpha})/c(h) < C, \ h>0, \tag{6}$$

then

$$\lim_{h\to\infty} (S_r f)(z+h)/c(h) = 0, \quad z \in \mathbb{C}\setminus(-\infty,0].$$

PROOF. (i) Since $f \in \mathcal{S}'_+$ there are $k,p \in \mathbb{N}_0$ and a locally integrable function F such that

$$f = D^k F, \ \operatorname{supp} F \subset [0,\infty) \ \text{and} \ \sup_x\{|F(x)|/(1+|x|^p)\} < \infty . \tag{7}$$

Take $r>p$. We have

$$(S_r f)(z) = (r+1)_k \int_0^\infty (F(t)/(z+t)^{r+k+1})dt, \ z \in \mathbb{C}\setminus\mathbb{R}. \tag{8}$$

On the other hand the assumption that the limit (5) exists (in $\mathcal{S}'$) implies that there is an m_0 such that for $m \geq m_0$

$$\lim_{h\to\infty} \left\langle \frac{f(t+h)}{c(h)}, \phi(t) \right\rangle = \langle \text{const.}, \phi \rangle, \quad \forall\phi \in S^m,$$

see [11, p.96, Corollary 2].

By taking sufficiently large m, say $m \geq m_1$, we have that

(9) $$\langle D^k F, \phi \rangle = \langle F, (-1)^k D^k \phi \rangle, \quad \forall\phi \in S^m,$$

holds in the sense of the dual pair (S'^m, S^m).

Let $m_o > p$, $m_1 \geq m_o$; put $r_o = m_1 + 1$. For $r > r_o$ we have that

(10) $$\mathbb{R} \ni t \to (z+t)^{r+1} \ (\mathrm{Im}z \neq 0) \text{ is from } S^m \text{ for } m > [m_1+1].$$

By (8), (9) and (10) we have

$$(S_r f)(z) = (r+1)_k \langle F(t), (z+t)^{-r-k-1} \rangle, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and

$$(S_r f)(z-h)/c(h) = (r+1)_k \langle F(t), (z-h+t)^{-r-k-1} \rangle / c(h) =$$

$$= \langle f(t), (z-h+t)^{-r-1} \rangle / c(h) = \langle f(t+h)/c(h), (z+t)^{-r-1} \rangle$$

$$\to \langle \text{const.}, (z+t)^{-r-1} \rangle = 0, \quad h \to \infty, \ \mathrm{Im}z \neq 0.$$

(ii) Let $\mathrm{Im}z \neq 0$. Since (6) implies that

$$f(t-h)/c(h) \to 0 \quad \text{in } S' \text{ as } h \to \infty,$$

by the same argument as in (i) we prove (ii).

Let $z = x \in (0,\infty)$ and let η be a smooth function such that $\eta(t) = 1$ for $t > -(x/2)$ and $\eta(t) = 0$ for $t < -x$. The function

(11) $$t \to \eta(t)/(x+t)^{r+1}, \quad t \in \mathbb{R},$$

is a smooth one.

Formally, we have

$$\left\langle f(t), \frac{\eta(t)}{(x+h+t)^{r+1}} \right\rangle = \left\langle f(t), \frac{\eta(t+h)}{(x+h+t)^{r+1}} \right\rangle =$$

$$= \left\langle f(t-h), \frac{\eta(t)}{(x+t)^{r+1}} \right\rangle.$$

By noting that $t \to \eta(t+h)/(t+h+x)^{r+1}$, $(h>0)$ is smooth for $t>0$ and repeating all the arguments as in (i) we can prove the assertion for $z=x \in (0,\infty)$.

REMARK. The arguments given above do not imply that

$(S_r f)(x-h)/c(h) \to 0$ as $h \to \infty$, $x \in (0,\infty)$.

Let $f \in S'_+$ and have the S-asymptotic at ∞ related to $h^\nu L(h)$ with $\nu>-1$. Then f has the quasiasymptotic at ∞ related to $k^\nu L(k)$ ([12]) and (*) characterizes completely the behaviour of $S_r f$ $(r>\nu)$ on the rays

$$z = \rho e^{i\phi}, \ \rho \in (0,\infty),$$

where ϕ is fixed and belongs to $(-\pi,\pi)$. If $L=1$ the second part of (*) gives the behaviour of $S_r f$ on the lines

$$z = x + iy, \ x \to \infty \ ,$$

where y is a fixed element of $\mathbb{R}$. This result is much more precise than the one which follows from Proposition 1. Note that (*) does not imply any result about the behaviour of $S_r f$ on the lines of the form

$$z = x + iy, \ x \to -\infty \ ,$$

where $\operatorname{Im} y \neq 0$.

3. Let $f(x)=H(x-1)/x^\alpha$, $x \in \mathbb{R}$, $\alpha>1$; f has the quasiasymptotic at ∞ related to k^{-1} with the limit δ and f has the S-asymptotic at ∞ related to $h^{-\alpha}$ with the limit 1. Proposition 1 implies that for suitable r

(12) $(S_r f)(z-h)/h^{-\alpha} \to 0$ as $h \to \infty$, $\operatorname{Im} z \neq 0$,

and (*) implies that

$$(S_r f)(z)/z^{-r-2} \to \text{const.}, \text{uniformly in } \Lambda_\varepsilon \text{ when } |z| \to \infty.$$

One can prove, by Lebesgue's theorem that for $\alpha=3$ and $r=1$ (12) holds. This implies that, in general case assertion (*) couldn't be extended on a domain which contains a line $x+iy$, $x \in (-\infty,0)$, with $y \neq 0$ fixed.

REFERENCES

[1] Lavoine,J., Misra,O.P., *Abelian Theorems for the Distributional Stieltjes Transformation, Proc.Camb. Phil. Soc., 86(1979), 287-293.*

[2] Nikolić-Despotović,D., Pilipović,S., *Tauberian Theorem for the Distributional Stieltjes Transformation, Internat.J.Math. Math.Sci., 9(1986),521-524.*

[3] Nikolić-Despotović,D., Pilipović,S., *Quasiasymptotic of Distributions at ±∞ and the Distributional Stieltjes Transformation, Rev.Res.Sci.Univ. Novi Sad, 16.2(1986), 41-52.*

[4] Pilipović,S., *Quasiasymptotic and the Distributional Stieltjes Transformation, Publ.Inst.Math. Beograd, 40(1986), 143-152.*

[5] Pilipović,S., *On the Quasiasymptotic of Schwartz's Distributions, Math.Nachr., 137(1988), 19-25.*

[6] Pilipović,S., *On the S-asymptotic of Temepred Distributios, Rev.Res.Sci.Univ. Novi Sad,Part I,Part II,15 n^o1(1985),47-58,59-67.*

[7] Pilipović,S., Stanković,B., *Initial Value Abelian Theorems for the Distributional Stieltjes Transform, Studia Math., LXXXVI (1987), 239-254.*

[8] Pilipović,S., Stanković,B., *Abelian Theorem for the Distributional Stieltjes Transform, Z.Anal. Anvend. 6(4)(1987), 341-349.*

[9] Pilipović,S., Stanković,B., *S-asymptotic of Distributions, Pliska (to appear).*

[10] Seneta,E., *Regularly Varying Functions, Lect.Not.Math. Springer, Berlin, New York, Heidelberg, 1976.*

[11] Vladimirov,V.S., *Generalized Functions in Mathematical Physics Mor., Moscow, 1979.*

[12] Vladimirov,V.S., Drožžinov,Yu.N., Zavialov,B.I., *Mulitdimensional Tauberian Theorems for Generalized Functions, Nauka, Moscow, 1986, (in Russian).*

REZIME

O DISTRIBUCIONOJ STIELTJESOVOJ
TRANSFORMACIJI I S-ASIMPTOTICI

Ispituje se ponašanje Stieltjesove transformacije distribucije koja ima odredjenu S-asimptotiku.

Received by the editors June 1,1986.