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AN EXPONENTIALLY FITTED QUADRATIC SPLINE DIFFERENCE SCHEME ON A NON-UNIFORM MESH

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Abstract

The uniformly convergent quadratic spline difference scheme is derived for the problem: $Ly = -\varepsilon y'' + q(x)y = f(x)$, $0 < x < 1$, $q(x) \geq q > 0$, $y(0) = \alpha_0$, $y(1) = \alpha_1$. A non-uniform mesh which provides for the location of a larger number of points in the boundary layers is used.

1. Introduced

An exponentially fitted quadric spline difference scheme for the problem:

$$(1)\qquad \begin{cases} Ly = -\varepsilon y'' + q(x)y = f(x), & 0 < x < 1, \\ y(0) = \alpha_0, \quad y(1) = \alpha_1, & q(x) \geq q > 0, \end{cases}$$

on a uniform mesh is derived in [3]. In this paper we shall generalize those result on a non-uniform mesh. Since spline difference schemes have the same order of accuracy and the same matrix structures on a uniform and

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a non-uniform grid, we wanted to transfer this characteristic on to an exponentially fitted quadratic spline difference scheme. But, there were some difficulties in determining the fitting factor. Namely, in [2] the fitting factor is determined from the requirement that the truncation error for the boundary layer functions should be equal to zero. Simultaneously, we obtain that the corresponding matrix has an iverse monotone form, and the obtained fitting factor approximates ε with the error $o(h^2)$. For this purpose, in the case of a non-uniform mesh, we must solve a system of 3 non-linear equations with 3 unknowns, which leads to very complicated and unuseful expressions.

So, we decided to put some conditions on the grid (natural for singular perturbation problems) in order to achieve a second order accuracy as on a uniform mesh.

2. Derivation of the scheme

Let us substitute differential equation (1) by the equation:

$$-\sigma(x)y'' + q(x)y = f(x) \ , \tag{2}$$

where $\sigma(x)$ is the fitting factor which will be determined subsequently. The approximate solution of equation (2) should be sought in the form of the quadratic spline $v(x) \in C^1[0,1]$:

$$v(x) = v_j + (x - x_j)v_j^{(1)} + \frac{(x - x_j)^2}{2} v_j^{(2)},$$

$$x \in [x_j,\ x_{j+1}],\quad 0 = x_0 < x_1 < \dots x_n < x_{n+1} = 1.$$

The points of collocation are points $x_{j+1/2} = x_j + h_j/2$, where $h_j = x_{j+1} - x_j$. Constants v_j, $v_j^{(1)}$ and $v_j^{(2)}$, $j=1(1)n$, are determined from the system of equations:

$$-\sigma_j v_j^{(2)} + q_{j+1/2}\left(v_j + \frac{h_j}{2} v_j^{(1)} + \frac{h_j^2}{8} v_j^{(2)}\right) = f_{j+1/2}$$

$$v_j = v_{j-1} + h_{j-1} v_{j-1}^{(1)} + \frac{h_{j-1}}{2} v_{j-1}^{(2)}$$

$$v_j^{(1)} = v_{j-1}^{(1)} + h_{j-1} v_{j-1}^{(1)}$$

By the elimination of $v_j^{(1)}$ and $v_j^{(2)}$ from the above equations, we get the difference scheme:

$$r^{-}v_{j-1} + r^{c}v_{j} + r^{+}v_{j+1} = q^{-}f_{j-1/2} + q^{+}f_{j+1/2} \tag{3}$$

$$r^{-} = -\frac{1}{b_{j-1}}\left(\frac{2\sigma_{j-1}}{h_{j-1}^{2}} - \frac{1}{4}q_{j-1/2}\right),$$

$$r^{+} = -\frac{1}{b_{j}}\left(\frac{2\sigma_{j}}{h_{j}^{2}} - \frac{1}{4}q_{j+1/2}\right),$$

$$r^{c} = -r^{-} - r^{+} + \frac{1}{b_{j}}q_{j+1/2} + \frac{1}{b_{j-1}}q_{j-1/2},$$

$$b_{j} = \frac{2\sigma_{j}}{h_{j}} + \frac{1}{4}q_{j+1/2}h_{j},$$

$$q^{-} = \frac{1}{b_{j-1}}, \qquad q^{+} = \frac{1}{b_{j}}.$$

In determining the fitting factor σ_j we use the following lemma:

Lemma 1. [1] *Let $y \in C^4[0,1]$. Let $q'(0) = q'(1) = 0$. Then the solution of problem (1) has the form:*

$$y(x) = u(x) + w(x) + g(x), \text{ where:}$$

$$u(x) = p_0 \exp\left(-x\sqrt{q(0)/\varepsilon}\right),$$

$$w(x) = p_1 \exp\left(-(1-x)\sqrt{q(1)/\varepsilon}\right),$$

p_0 and p_1 are bounded functions of ε independent of x and:

$$|g^{(j)}| \le M\,(1 + \varepsilon^{1-(1/2)j}), \quad j=0(1)4$$

M is a constant independent of ε.

The local truncation error of scheme (3) for an arbitrary smooth function φ has the form:

$$\tau_j(\varphi) = R_h\varphi_j - Q_h(L\varphi)_j \quad \text{where:}$$

$$R_h v_j = r_j^{-}v_{j-1} + r_j^{c}v_j + r_j^{+}v_{j+1},$$

$$Q_h f_j = q^{-}f_{j-1/2} + q^{+}f_{j+1/2}.$$

Lemma 2. *Let, in scheme (3), σ_{j-1} be replaced by σ_j^{-} and σ_j by σ_j^{+}, where:*

$$\sigma_j^{-} = h_{j-1}^{2}\, q_{j-1/2}\, S_j(q_{j-1/2})/8,$$

$$\sigma_j^{+} = h_{j}^{2}\, q_{j+1/2}\, S_j(q_{j+1/2})/8,$$

$$S_j(t) = 1 + \frac{4(h_j + h_{j-1})}{s_j(t)},$$

$$s_j(t) = h_j \exp(h_{j-1} \sqrt{t/\varepsilon}) - (h_j + h_{j-1}) +$$

$$h_{j-1} \exp(-h_j \sqrt{t/\varepsilon}) \quad for \quad j = 0(1)i_1 \quad and$$

$$s_j(t) = h_j \exp(-h_{j-1} \sqrt{t/\varepsilon}) - (h_j + h_{j-1}) +$$

$$h_{j-1} \exp(h_j \sqrt{t/\varepsilon}) \quad for \quad j = i_2(1)n+1, \quad i_1 \le i_2 .$$

For $i_1 < j < i_2$ in the previous expressions we put $h_j = h = \text{const}$. Then:

a) $\tau_j(u) = 0,\ \ j = 0(1)i_1$ *for* $q(x) = q = \text{const}$.

b) $\tau_j(w) = 0,\ \ j = i_2(1)n+1$ *for* $q(x) = q = \text{const}$.

c) $\tau_j(u) = \tau_j(w) = 0$ *for* $i_1 < j < i_2$.

d) The matrix of system (3) is an inverse monotone.

Proof. a) By the direct substitution of σ_j and σ_{j-1} in the expression $R_h u_j$, we obtain zero. Since $Lu_j = 0$, we have that $\tau_j(u) = 0$.

In the same way we obtain that *b*) and *c*) hold.

d) Since $r^- \le 0$, $r^+ \le 0$, $r^c > (-r^- + r^+)$ the proof follows.

Throughout the paper M and δ denote different constants independent of ε and h_j.

Lemma 3. *Let:*

$$(4) \qquad \begin{cases} 0 \le h_j - h_{j-1} \le M_j \dfrac{h_{j-1}}{\varepsilon} \min(M_0 h_{j-1}^2, \varepsilon), & 1 \le j \le i_1 , \\[2ex] 0 \le h_{j-1} - h_j \le M_j \dfrac{h_j}{\varepsilon} \min(\bar{M}_0 h_j^2, \varepsilon), & i_2 \le j \le n+1 , \end{cases}$$

$h_j/h_{j\pm1} \le M$, $h_j = h = \text{const.}$, $i_1 < j < i_2$,

$M_i \le M$, $i = 0(1)n+1$, $\bar{M}_0 < M$.

Then:

$\max(|\sigma_j^- - \varepsilon|,\ |\sigma_j^+ - \varepsilon|) \le Mh_j^2$, $j = 1(1)n+1$.

Proof. Let $\varepsilon \le M_0 h_j^2$. Then:

$$\left|\frac{h_j + h_{j-1}}{s_j(t)}\right| \le M \exp(-h_{j-1} \sqrt{t/\varepsilon}) , \qquad |S_j(t)| \le M ,$$

$$\max(|\sigma_j^+|,\ |\sigma_j^-|) \le Mh_j^2 \quad \textit{and} \quad |\sigma_j^\pm - \varepsilon| \le Mh_j^2 .$$

Let $\varepsilon \ge M_0 h_j^2$. After some Taylor expansions we have:

$$\frac{4(h_j+h_{j-1})}{s_j(q_{j-1/2})} = \frac{8}{h_j h_{j-1} q_{j/1/2}} + 2\,\frac{h_j-h_{j-1}}{h_j h_{j-1}}\sqrt{\frac{\varepsilon}{q_{j-1/2}}} + N ,$$

$|N| \le M$, and from (4) we obtain $|\sigma_j^- - \varepsilon| \le Mh_j^2$. In the same manner we obtain $|\sigma_j^+ - \varepsilon| \le Mh_j^2$.

3. Proof of uniform convergence

Lemma 4. *Let* $y(x) \in C^4[0,1]$ and $q'(0) = q'(1) = 0$. *Let* σ_j *and* σ_{j-1} *in* (3) *be determined by Lemma 2. Let Lemma 3 hold. Then:*

$$|\tau_j(y)| \le \begin{cases} M\dfrac{h_j^3}{\varepsilon}, & M_0 h_{j-1}^2 \le \varepsilon \ \text{ or } \ h_j^2 \bar{M}_0 \le \varepsilon , \\[2ex] Mh_j, & M_0 h_j^2 \ge \varepsilon \ \text{ or } \ h_j^2 \bar{M}_0 \ge \varepsilon , \end{cases}$$

Proof. According to Lemma 1 we have:

$$\tau_j(y) = \tau_j(u) + \tau_j(w) + \tau_j(g) .$$

We shall estimate separately the truncation error for each function u, w, and g. We shall start with $u = u(x)$. Denote by $\tilde{\tau}_j(u)$ the truncation error $\tau_j(u)$ in the case $q(x) = q_0 = q(0) = \text{const}$. Since $\tilde{\tau}_j(u) = 0$, we have that:

$$\tau_j(u) = \tau_j(u) - \tilde{\tau}_j(u), \quad j = 0(1)i_2-1. \tag{5}$$

After some Taylor developments we have:

$$\tau_j(y) = T_2 y_j'' + T_3 y_j''' + \frac{r^- h_{j-1}^4}{4!} y^{IV}(\xi_1) + \frac{r^+ h_j^4}{4!} y^{IV}(\xi_2) +$$
$$\varepsilon\left[q^+ \frac{h_j^2}{8} y^{IV}(\xi_3) - q^- \frac{h_{j-1}^2}{8} y^{IV}(\xi_4)\right] + q^- \frac{h_{j-1}^4}{384} y^{IV}(\xi_5) -$$
$$q^+ \frac{h_j^4}{384} y^{IV}(\xi_6), \quad x_{j-1} \le \xi_i \le x_{j+1}, \quad i = 1(1)6. \tag{6}$$

$$T_2 = r^- \frac{h^2_{j-1}}{2} + r^+ \frac{h^2_j}{2} + \varepsilon\,(q^+ + q^-) - q^- \frac{h^2_{j-1}}{8} - q^+ \frac{h^2_j}{8},$$

$$T_3 = -\frac{h^3_{j-1}}{3!} r^- + \frac{h^3_j}{3!} r^+ + \varepsilon(-q^- \frac{h_{j-1}}{2} + q^+ \frac{h_j}{2}) + q^- q_{j-1/2} \frac{h^3_{j-1}}{48} -$$

$$q^+ q_{j+1/2} \frac{h^3_j}{48}.$$ Let $h_{j-1} M_0 \le \varepsilon$.

Taking into account (6) and (5), we obtain that:

$$|\tau_j(u)| \le M \frac{q(x_j)-q(0)}{\varepsilon^2} h^3_j u_j \le M \frac{h^3_j}{\varepsilon}, \quad j = 1(1)i_2-1.$$

According to Lemma 1, from (6) and (4) we have:

$$\max\,(|\tau_j(w)|,\ |\tau_j(g)|) \le Mh^3_j/\varepsilon, \quad j = 1(1)i_2-1.$$

For $i_2 \le j \le n+1$ and $\bar{M}_0 h^2_j \le \varepsilon$ we have:

$$|\tau_j(w)| = |\tau_j(w) - \tilde{\tau}_j(w)| \le M \frac{q(x_j)-q(0)}{\varepsilon^2} h^3_j v_j \le Mh^3_j/\varepsilon,$$

$\max(|\tau_j(u)|, |\tau_j(g)| \le Mh^3_j/\varepsilon$. Thus, Lemma 4 holds for $h_{j-1} M_0 \le \varepsilon$, or $h_j \bar{M}_0 \le \varepsilon$.

In the opposite case we use a truncation error in the form:

$$\text{(7)}\qquad \tau_j(y) = r^- \frac{h^2_{j-1}}{2!} y''(\xi_1) + \frac{r^+ h^2_j}{2!} y''(\xi_2) + \varepsilon(q^- y''_{j-1/2} +$$

$$q^+ y''_{j+1/2}) - q^- \frac{h^2_{j-1}}{8} y''(\xi_3) - q^+ \frac{h^2_j}{8} y''(\xi_4),$$

$$x_{j-1} \le \xi_i \le x_{j+1}, \quad i = 1(1)4.$$

Since $|r^\mp| \le Mh_j^{-1}$ and $|q^\mp| \le M_j^{-1}$, we have that:

$$\max\,(|\tau_j(w)|,\ |\tau_j(u)|) \le Mh_j, \quad j = 1(1)i_2-1,$$

and from $\tau_j(u) = \tau_j(u) - \tilde{\tau}_j(u)$ we obtain that the same holds for function u.

For $i_2 \le j \le n+1$ we have that $\tau_j(w) = \tau_j(w) - \tilde{\tau}_j(w)$ and from (7) we can conclude that:

$\max(|\tau_j(w)|,\ |\tau_j(u)|,\ |\tau_j(g)|) \le Mh_j$. Thus, Lemma 4 holds.

Theorem 1. *Let $y(x) \in C^4[0,1]$. Let $q'(0) = q'(1) = 0$. Let the mesh points satisfy condition (4). Let v_j, $j = 0(1)n+1$, be the approximation to the solution $y(x_j)$ obtained by using (3).*

Then:

$|y(x_j)-v_j| \le Mh_j^2$, $j=0(1)n+1$, *where M is a constant independent of ε and h_j.*

Proof. Since:

$$r^- + r^c + r^+ \ge \begin{cases} M\varepsilon/h_j, & \text{for } M_0 h_{j-1} \le \varepsilon \text{ or } \bar{M}_0 h_j \le \varepsilon \\ Mh_j, & \text{for } M_0 h_{j-1} \ge \varepsilon \text{ or } \bar{M}_0 h_j \ge \varepsilon, \end{cases}$$

from the inequalities:

$R_h[\pm(y(x_j) - v_j) + Mh^{-2}] \ge 0$, $\bar{h} = \max(h_j, h_{j-1}) = \text{const}$. we obtain the statement of Theorem 1 (the inverse monotony of the corresponding matrix and Lemma 4).

Remark 1. For $h_j = h = \text{const}$, $j = 0(1)n$, scheme (3) reduces to the one given in [3].

For $\sigma_j = \varepsilon = 1$ scheme (3) reduces to the one corresponding to the collocation spline given in [2].

5. Numerical results

Scheme (3) was used to obtain the approximate solution of problem [1]:

$$-\varepsilon y'' + (1 + x(1-x))y = f(x),$$

$$y(0) = y(1) = 0.$$

Its exact solution is:

$$y(x) = 1 - (1-x)\exp(-x/\sqrt{\varepsilon}) - x\exp((x-1)/\sqrt{\varepsilon}).$$

Here, we shall present the numerical results which suggest the choice of the grid.

Tables 1, 2 and 3 contain the maximal differences between the exact and approximate solution at the points of the grid, for different ε and n.

Tabel 1 contains the results on an equidistant grid.

Table 1.

$\max_j |v_j - v(x_j)|$

$n+1$ / ε	32	64	128	256	512
1/2	0.17-04	0.42-05	0.10-05	0.26-06	0.66-07
1/4	0.29-04	0.73-05	0.18-05	0.43-06	0.11-06
1/8	0.40-04	0.10-04	0.25-05	0.62-06	0.16-06
1/16	0.43-04	0.11-04	0.27-05	0.67-06	0.17-06
1/32	0.49-04	0.12-04	0.31-05	0.77-06	0.19-06
1/64	0.66-04	0.16-04	0.41-05	0.10-05	0.26-06
1/128	0.93-04	0.23-04	0.57-05	0.14-05	0.36-06
1/256	0.12-03	0.33-04	0.81-05	0.20-05	0.51-06
1/512	0.19-03	0.46-04	0.11-04	0.24-05	0.72-06
1/1024	0.26-03	0.63-04	0.16-04	0.41-05	0.10-05

A non-equidistant grid is formed in such a way that more of points are found in layers than outside them.

On the interval $[0,c_0]$, c_0 is a constant which is given in advance, the grid is non-equidistant and obtained according to the formula:

$$\tilde{h}_j = \tilde{h}_{j-1} + M\frac{\tilde{h}_{j-1}}{\varepsilon}\min(\tilde{h}^2_{j-1}, \varepsilon), \qquad j = 1(1)n_1-1,$$

$$h_j = Q\tilde{h}_j, \quad Q = c_0/\tilde{Q}, \quad \tilde{Q} = \sum_{j=1}^{n_1-1}\tilde{h}_j, \quad x_j = x_{j-1} + h_{j-1}, \quad j = 1(1)n_1,$$

$x_0 = 0$, $\tilde{h}_0$, M and c_0 are given, $M_0 = \bar{M}_0 = Q^{-2}$.

On the interval $[c_0, 1-c_0]$, the grid is equidistant:

$$h_j = (1-2c)/n_2, \qquad j = n_1(1)n_1+n_2-1,$$

$$n+1 = 2n_1 + n_2, \quad x_j = x_{j-1} + h, \quad j = n_1+1, \quad n_1 + n_2-1.$$

On the interval $[1-c_0, 1]$, the grid is symmetric to the grid on interval $[0,c_0]$. This is a starting mesh. In each next spet the intervals are halved.

Table 2

$c_0 = 0.05$, $h_0 = 0.01$, $M=46$, $n_1(n+1) = 3/16$

$\max_j |v_j - y(x_j)|$

ε \ $n+1$	32	64	128	256	512
1/2	0.21-04	0.54-05	0.14-05	0.34-06	0.86-07
1/4	0.59-04	0.15-04	0.37-05	0.94-06	0.24-06
1/8	0.99-04	0.25-04	0.64-05	0.16-05	0.40-06
1/16	0.15-03	0.39-04	0.99-05	0.25-05	0.63-06
1/32	0.25-03	0.67-04	0.17-04	0.44-05	0.11-05
1/64	0.38-03	0.10-03	0.26-04	0.67-05	0.17-05
1/128	0.50-03	0.13-03	0.35-04	0.89-05	0.22-05
1/256	0.57-03	0.15-03	0.38-04	0.96-05	0.24-05
1/512	0.52-03	0.12-03	0.31-04	0.79-05	0.20-05
1/1024	0.38-03	0.78-04	0.18-04	0.45-05	0.11-05
1/2048	0.28-03	0.77-04	0.15-04	0.50-05	0.13-05
1/216	0.43-03	0.25-03	0.11-03	0.41-04	0.13-04

(37% points in the layers)

Table 3

$c_0 = 0.05$, $h_0 = 0.01$, $M=7,5$, $n_1/(n+1) = 5/16$

$\max_j |v_j - y(x_j)|$

ε \ $n+1$	32	64	128	256	512
1/2	0.55-04	0.14-04	0.37-05	0.93-06	0.23-06
1/4	0.16-03	0.41-04	0.10-04	0.26-05	0.66-06
1/8	0.27-03	0.70-04	0.18-04	0.45-05	0.11-05
1/16	0.39-03	0.11-03	0.28-04	0.70-05	0.12-05
1/32	0.64-03	0.18-03	0.47-04	0.12-04	0.31-05
1/64	0.91-03	0.26-03	0.71-04	0.18-04	0.47-05
1/128	0.11-02	0.33-03	0.92-04	0.24-04	0.62-05
1/256	0.11-02	0.35-03	0.99-04	0.26-04	0.69-05
1/512	0.12-02	0.31-03	0.84-04	0.22-04	0.58-05
1/1024	0.78-03	0.18-03	0.44-04	0.11-04	0.29-05
1/216	0.23-03	0.75-04	0.22-04	0.64-05	0.18-05
1/219	0.64-04	0.22-04	0.71-05	0.36-05	0.14-05

(62,5% points in the layers)

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Rezime

EKSPONENCIJALNO FITOVANA SPLAJN DIFERENCNA ŠEMA SA KVADRATNIM SPLAJNOVIMA NA NERAVNOMERNOJ MREŽI

Izvedena je uniformno konvergentna splajn diferencna šema sa kvadratnim splajnovima za problem $Ly = -\varepsilon y'' + q(x)y = f(x)$, $0 < x < 1$, $q(x) \geq q > 0$, $y(0) = \alpha_0$, $y(1) = \alpha_1$. Korišćena je neravnomerna mreža koja omogućuje smeštanje većeg broja tačaka u granične slojeve.

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