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S-ASYMPTOTIC OF SOLUTIONS OF THE ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

Bogoljub Stanković

University of Novi Sad, Faculty of Science, Institute of Mathematics, Dr I. Djuričića 4, 21000 Novi Sad Yugoslavia

ABSTRACT

The S-asymptotic behaviour of solutions of non-homogeneous elliptic partial differential equations is given on a ray $\{\rho w, \rho>0\}$, where $w \in R^n$ and $\|w\|=1$.

INTRODUCTION.

The S-asymptotic is one of the notions related to the asymptotical behaviour of distributions elaborated in [3] . It can be profitable in many studies and applications (see [5]), especially in the quantum field theory. In [1] one can find references to papers from the quantum field theory which pushed forward the study of the S-asymptotic behaviour of distributions.

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1. FUNDAMENTAL SOLUTIONS OF AN ELLIPTIC DIFFERENTIAL EQUATION

A fundamental solution of a differential equation

$$P(i \frac{\partial}{\partial x})u=0$$

in n-dimensions is a distribution $G \in D'(R^n)$ satisfying the equation

$$P(i \frac{\partial}{\partial x})G(x)=\delta(x).$$

Let $P_o(i \frac{\partial}{\partial x})$ be the principal part of $P(i \frac{\partial}{\partial x})$. If $P_o(y)\neq 0$ for any real $y\neq 0$, then $P(i \frac{\partial}{\partial x})$ is called an elliptic operator. Assume that the coefficients of P are real numbers. If the elliptic operator is also homogeneous (i.e. if it coincides with its principal part), then fundamental solution G(x) has the form:

$$(1) \quad G(x)= \begin{cases} A(\frac{x}{r})r^{m-n}, & \text{for n odd, or n even and } m<n \\ B(\frac{x}{r})r^{m-n}+C(x)\ \ln r, & \text{for n even and } m\geq n\ 1. \end{cases}$$

where A(y) and B(y) are analytic functions on $\|y\|=1$ and C(x) is a polynomial in x of degree m-n ; m is the degree of the principal polynomial $P_o(y)$ (m is then necessarily an even number) [2]; $r^2=\|x\|^2= \sum_{i=1}^{n} x_i^2$.

2. S-ASYMPTOTIC OF FUNDAMENTAL SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

We shall use the following definition of the S-asymptotic in S'[3] :

<u>Definition.</u> Distribution $T \in S'$ has the S-asymptotic related to $c(h)>0$ in a cone Γ with the limit $U \in S'$ if and only if there exists the limit:

$$(2) \quad \lim_{h\in\Gamma, \|h\|\to\infty} \langle T(x+h)/c(h), \varphi(x)\rangle = \langle U,\varphi\rangle$$

for every $\varphi \in S$. If $\Gamma \equiv \{\rho w, \rho>0\}, \|w\|=1$, we write in short:

$$T(x+\rho w) \overset{S}{\sim} c(\rho)U \ , \ \rho\to\infty.$$

Our aim it to find S-asymptotic of the fundamental solution G given by relation (1).

First we shall treat the function $A(\frac{x}{r})\ r^{m-n}$, $m>n$; $A(y)$ is analytic on $\|y\|=1$. Γ will be the ray $\{\rho w, \rho>0\}$ for a fixed $w \in R^n$, $\|w\|=1$. In this case limit (2) is

$$J \equiv \lim_{\rho\to\infty} \langle \rho^{n-m} A((x+\rho w)/\|x+\rho w\|)\|x+\rho w\|^{m-n}, \varphi(x)\rangle$$

$$= \lim_{\rho\to\infty} \int_{R^n} A((\frac{x}{\rho}+w)/\|\frac{x}{\rho}+w\|)\|\frac{x}{\rho}+w\|^{m-n}\varphi(x)\ dx\ .$$

Since

$$A((\frac{x}{\rho}+w)/\|\frac{x}{\rho}+w\|)\|\frac{x}{\rho}+w\|^{m-n} <$$

$$< M\,[\sum_{i=1}^{n}(|x_i|+|w_i|)^2]^{(m-n)/2}\ ,\ \rho>1$$

where $M=\sup A(y)$, $\|y\|=1$, we can use the Labegue's Lemma which gives

$$J=A(w/\|w\|)\|w\|^{m-n}\int_{R^n}\varphi(x)\ dx = A(w)\int_{R^n}\varphi(x)\ dx.$$

We proved that $A(\frac{x}{r})\ r^{m-n}$, $m>n$, has the S-asymptotic related to ρ^{m-n} on the ray $\{\rho w, \rho>0\}$, $\|w\|=1$ and with the limit $U=A(w)$.

The next step is to find the S-asymptotic of the function $A(\frac{x}{r})\ r^{m-n}$, but in case $m<n$, $m>0$.

Suppose that in relation (2) the function φ belongs to D and supp $\varphi \subset I^n$, $I=(-a,a), a \in R_+$. Then for $\rho > a/\max|w_i|$, $i=1,\dots,n$

$$\lim_{\rho\to\infty} \rho^{n-m}\int_{R^n} A((x+\rho w)/\|x+\rho w\|)\|x+\rho w\|^{m-n}\varphi(x)\ dx$$

$$=A(w)\int_{R^n}\varphi(x)\ dx.$$

Hence, $A(\frac{x}{r})r^{m-n}$, $m<n$, has the S-asymptotic in D' with the limit $A(w)$. Since D is dense in S, to prove that this func-

tion has the same S-asymptotic in S', it suffices to prove that $\rho^{n-m}J(\rho)$, $\rho>0$, is bounded (see the Banach-Steinchaus theorem), where

$$J(\rho)=\int_{R^n} A\left(\frac{x+\rho w}{\|x+\rho w\|}\right)\|x+\rho w\|^{m-n}\varphi(x)\,dx\ ,\quad \varphi\in S.$$

Therefore, we shall divide the integral $J(\rho)$ into three parts and use the following inequalities:

For $\|x\|<\rho/2$ and $\|x\|\geq 3\rho/2$, $\|x+\rho w\|\geq|\|x\|-\rho|\geq\rho/2$. For $\varphi\in S$ and $\rho/2\leq\|x\|\leq 3\rho/2$, $|\varphi(x)|\leq K(1+\|x\|^2)^{-k}\leq K(1+(\rho/2)^2)^{-k}$, where k is any integer and K is a constant.

Now, we have for $\varphi\in S$, $m<n$, $m>0$:

$$\rho^{n-m}\left|\int_{\|x\|<\frac{\rho}{2}}+\int_{\|x\|>\frac{3\rho}{2}} A\left(\frac{x+\rho w}{\|x+\rho w\|}\right)\|x+\rho w\|^{m-n}\varphi(x)\,dx\right|\leq$$

$$\leq 2\ 2^{n-m}\,M\int_{R^n}|\varphi(x)|\,dx,$$

where $M=\sup|A(w)|$, $\|w\|=1$, and

$$\left|\int_{\|x\|\geq\frac{\rho}{2}}^{\|x\|<\frac{3\rho}{2}} A\left(\frac{x+\rho w}{\|x+\rho w\|}\right)\|x+\rho w\|^{m-n}\varphi(x)\,dx\right|\leq$$

$$\leq K\frac{M}{(1+\frac{\rho^2}{4})^k}\int_{\|x\|\geq\frac{\rho}{2}}^{\|x\|<\frac{3\rho}{2}}\|x+\rho w\|^{m-n}\,dx\to 0,\ \rho\to\infty.$$

Assume, now, that $C(x) = \sum_{|i|\leq m-n} \alpha_i x^i$ $m\geq n$, where $\alpha_i \in R$; $i=(i_1,\dots,i_n)$; $x^i = x_1^{i_1} \dots x_n^{i_n}$; $|i|=i_1+\dots i_n$. Then,

$$\lim_{\rho\to\infty} \rho^{n-m}(\ln\rho)^{-1}\langle C(x+\rho w)\ \ln\|x+w\|, \varphi(x)\rangle =$$

$$= \lim_{\rho\to\infty} \rho^{n-m}(\ln\rho)^{-1} \int_{R^n} \sum_{|i|\leq m-n} \alpha_i\ (x+\rho w)^i (\ln\rho +$$

$$+ \ln\|\frac{x}{\rho} + w\|)\varphi(x)\ dx = \sum_{|i|=m-n} \alpha_i\ w^i \int_{R^n}\varphi(x)\ dx$$

by the same reason as in the first case.

The results of this paragraph can be expressed in the following

Proposition 1. If G is a fundamental solution of the eliptic homogeneous operator P of degree m in n--dimension, then on the ray $\{\rho w, \rho>0\}, \|w\|=1$, G has an S-asymptotic:

$G(x+\rho w) \overset{s}{\sim} \rho^{m-n}\cdot G(w)$, $\rho\to\infty$ for n odd, or n even and $m<n$;

(3) $G(x+\rho w) \overset{s}{\sim} \rho^{m-n}\ln\rho\cdot D(w)$, $\rho\to\infty$, for n even and $m\geq n$,

where $D(w) = \lim_{\rho\to\infty} G(\rho w)/(\rho^{m-n}\ln\rho)$.

2. S-ASYMPTOTIC OF SOLUTIONS OF A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

Now, we can give the S-asymptotic behaviour of solutions of the partial differential equation:

(4) $P(i\frac{\partial}{\partial x})u(x)=f(x)$, $f\in \mathcal{O}_c'$

where P is an elliptic homogeneous operator of degree m in n-dimensions. We know that a solution of equation (4) is of the form $u=G*f$ (G being the fundamental solution of the operator P) and belongs to S'. Namely, since G belongs to S' and f to O'_C, then $G*f$ exists and belongs to S' [4]. Moreover the mapping: $(G,f)\ G*f$ is separately continuous.

<u>Proposition 2.</u> Equation (4) has a solution $u=G*f$ belonging to S'. u has an S-asymptotic on the ray $\{\rho w, \rho>0\}, \|w\|=1, \rho\to\infty$:

$$u(x+\rho w) \overset{s}{\sim} \rho^{m-n}\cdot(G(w)*f), \text{ for n odd, or n even and } m<n$$

(5) $$u(x+\rho w) \overset{s}{\sim} \rho^{m-n}\ln\rho\cdot(D(w)*f), \text{ for n even and } m\geq n,$$

where G is the fundamental solution of the elliptic differential equation $P(i\frac{\partial}{\partial x})G(x)=\delta(x)$ and $D(w)=\lim_{\rho\to\infty} G(\rho w)/(\rho^{m-n}\ln\rho)$. P is a homogeneous operator of degree m in n-dimensions

<u>Proof.</u> Suppose that G is a fundamental solution of the equation $P(i\frac{\partial}{\partial x})u(x)=\delta(x)$, then G has the form given in (1) and has the S-asymptotic given by relation (3). A solution of equation (4) is $u=G*f$. Since this convolution is separately continuous, from (3) follows (5).

<u>REMARK</u>. It is well known that if v is a solution of equation (4) and u-v has the convolution with G, then v=u. This fact says about the class of distributions in which our solution $u=G*f$ is unique.

If P is elliptic but not at any rate homogeneous, then the fundamental solution G(x) of (4) has the from

$$A(\frac{x}{r},r)r^{m-n} \text{ , for n odd}$$

$$B(\frac{x}{r},r)r^{m-n}+C(\frac{x}{r},r)r^{m-n}\ln r, \text{ for n even,}$$

where A(w,r), B(w,r), C(w,r) are analytic functions in (w,r) in a neighbourhood of $\|w\|=1$, $r=0$, and $C(\frac{x}{r},r)r^{m-n}$ is a function C(x) which is analytic in x at x=0 (see [2]).

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REZIME

S-ASIMPTOTIKA REŠENJA PARCIJALNE DIFERENCIJALNE JEDNAČINE ELIPTIČKOG TIPA

U radu je dokazano sledeće tvrdjenje:

Jednačina (4) ima rešenje $u=G*f$ koja pripada skupu S'. To rešenje u ima nad zrakom $\{\rho w, \rho>0\}, \|w\|=1$ S-asimptotiku kada $\rho\to\infty$:

$$u(x+\rho w) \overset{s}{\sim} \rho^{m-n}\cdot(G(w)*f), \quad \text{za } n \text{ neparno, ili } n \text{ parno i } m<n$$

$$u(x+\rho w) \overset{s}{\sim} \rho^{m-n}\ln\rho\cdot(D(w)*f), \quad \text{za } n \text{ parno i } m\geq n$$

gde je G fundamentalno rešenje jednačine (4), a $D(w)= = \lim_{\rho\to\infty} G(\rho w)/(\rho^{m-n}\ln\rho)$.

Jedinstvenost rešenja u je u klasi distribucija v koje imaju osobinu da $u-v$ ima konvoluciju sa G.

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