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A REMARK CONCERNING SLOWLY VARYING FUNCTIONS IN KARAMATA'S SENSE

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ABSTRACT

In this paper the author give a complete proof of a statement from the paper [1] .

1. INTRODUCTION

In [1] we were concerned with the properties of slowly varying functions in Karamata's sense i.e. of real functions L defined and positive for $x>0$, such that

$$\lim_{x\to+\infty} \frac{L(\lambda x)}{L(x)} = 1 \quad \text{for any } \lambda\in(0,+\infty).$$

In paper [3] J. Karamata defined this class of functions, and established also, using a particular example (p. 46-47), that a slowly varying function need not have the asymptotic behaviour of a monotonic function as $x\to+\infty$. In [1] we made this fact more precise by assertion 4^0 of Theorem II, by which: a slowly varying function may,

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as $x\to+\infty$, oscillate, with a finite or infinite interval of oscillation ([1], p. 133). However, proving this statement, we only constructed ([1], p. 134-135) a definite slowly varying function which oscillates between finite (and positive) limits as $x\to+\infty$, and noted finally that "one can, by suitable modifications of this example, construct a slowly varying function with an infinite interval of oscillation".

Here we shall give a complete proof of the second part of the cited statement, which has not been made in [1], and we shall also give a more precise form of this statement. Namely, we shall replace it by the following:

Proposition. For any two elements a and b of the interval $[0,+\infty]$ such that $a<b$, there exists a slowly varying function L continuous on $[0,+\infty)$ and such that

$$\underline{\lim}_{x\to+\infty} L(x)=a, \qquad \overline{\lim}_{x\to+\infty} L(x)=b.$$

Proof. In all cases considered, the slowly varying function $L(x)$ which we shall construct for $x\geq 1$ will be given by an equality of the form

$$L(x)=e^{\int_1^x \frac{\varepsilon(t)}{t}\,dt},$$

where

$$\varepsilon(x)=\frac{\eta(x)}{1+\ln x} \quad (x\geq 1) \tag{1}$$

and the function $\eta(x)$ is continuous and bounded for $x\geq 1$; for $x\in[0,1]$ it is sufficient to define $L(x)$ as a function continuous and positive on that interval. According to the known result about the representation of a slowly varying function (see, for example, [1], p.124, 0.2), the function $L(x)$ defined in this way is certainly slowly varying and continuous on $[0,+\infty)$.

We shall first consider the case

$$0<a<b<+\infty.$$

Let us put

(2) $\rho=\ln b-\ln a\ (>0),\quad \omega=e^{e^{\rho}-1}-1\,(>0),\quad \delta=\frac{\omega}{3}\ (>0)$

and

(3) $x_1=1+2\delta;\quad x_{n+1}=\delta+e^{e^{\rho}-1}\cdot(\delta+x_n)^{e^{\rho}}\ (n\in N).$

Then,

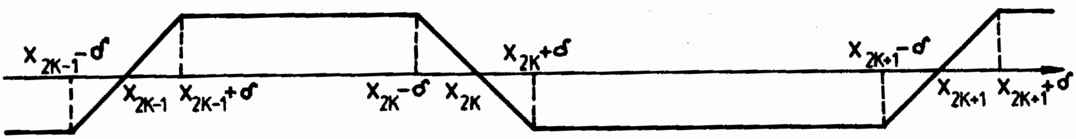
$$\frac{x_{n+1}}{x_n}>e^{e^{\rho}-1}\cdot\frac{(\delta+x_n)^{e^{\rho}}}{x_n}>e^{e^{\rho}-1};\quad \frac{\delta+x_n}{x_n}>e^{e^{\rho}-1}=\omega+1\,(>1)\ (n\in N)$$

and hence,

(4) $x_{n+1}-x_n=x_n\left(\frac{x_{n+1}}{x_n}-1\right)>x_1\omega>\omega=3\delta\ (n\in N).$

Let us further pose (see the figure)

(5)
$$\eta(x)=\begin{cases} x-x_{2k-1} & (x_{2k-1}-\delta\le x<x_{2k-1}+\delta)\\ 1 & (x_{2k-1}+\delta\le x<x_{2k}-\delta)\\ -x+x_{2k} & (x_{2k}-\delta\le x<x_{2k}+\delta)\\ -1 & (x_{2k}+\delta\le x\le x_{2k+1}-\delta)\end{cases}\quad (k\in N).$$



According to (2), (3) and (4), this definition of $\eta(x)$ is consistent and obviously $\eta(x)$ is defined by it for $x\ge x_1-\delta$ (>1) as a function continuous and bounded on $[0,+\infty)$. For $1\le x\le x_1-\delta$, we define $\eta(x)$ as a function continuous on $[1,x_1-\delta]$ and such that

$$\int_1^{x_1-\delta}\frac{\eta(t)}{t(1+\ln t)}\,dt=\ln a-\alpha-\int_{x_1-\delta}^{x_1}\frac{t-x}{t(1+\ln t)}\,dt,$$

where the number α will be determined later; evidently, this can be done.

By (3), we have

$$\int_{x_k+\delta}^{x_{k+1}-\delta}\frac{dt}{t(1+\ln t)}=\ln(1+\ln t)\Big|_{x_k+\delta}^{x_{k+1}-\delta}=\ln\frac{1+\ln(x_{k+1}-\delta)}{1+\ln(x_k+\delta)}$$

$$=\ln\frac{1+\ln[e^{e^{\rho}-1}(\delta+x_k)^{e^{\rho}}]}{1+\ln(x_k+\delta)}=\ln\frac{1+e^{\rho}-1+e^{\rho}\ln(\delta+x_k)}{1+\ln(x_k+\delta)}=\rho.$$

Hence,

$$\int_1^{x_{n+1}}\frac{\varepsilon(t)}{t}\,dt=\int_1^{x_1-\delta}\frac{\eta(t)}{t(1+\ln t)}\,dt+\int_{x_1-\delta}^{x_1}\frac{t-x_1}{t(1+\ln t)}\,dt+\sum_{k=1}^{n}\int_{x_k}^{x_{k+1}}\frac{\varepsilon(t)}{t}\,dt$$

$$=\ln a-\alpha+\sum_{k=1}^{n}(-1)^{k-1}\int_{x_k}^{x_{k+1}}\frac{|\eta(t)|}{t(1+\ln t)}\,dt$$

$$=\ln a-\alpha+\sum_{k=1}^{n}(-1)^{k-1}\Big[\int_{x_k+\delta}^{x_{k+1}-\delta}\frac{dt}{t(1+\ln t)}+\Big(\int_{x_k}^{x_k+\delta}+\int_{x_{k+1}-\delta}^{x_{k+1}}\Big)\frac{|\eta(t)|}{t(1+\ln t)}\,dt\Big]$$

$$=\ln a-\alpha+\sum_{k=1}^{n}(-1)^{k-1}(\rho+\alpha_k),$$

where

$$\alpha_n\overset{\text{def}}{=}\Big(\int_{x_n}^{x_n+\delta}+\int_{x_{n+1}-\delta}^{x_{n+1}}\Big)\frac{|\eta(t)|}{t(1+\ln t)}\,dt\downarrow 0\quad(n\to\infty),$$

and consequently

$$\alpha \overset{\text{def}}{=\!=} \sum_{k=1}^{\infty}(-1)^{k-1}\alpha_k$$

is a finite number. Hence,

$$(6)\quad \int_1^{x_{2n+1}} \frac{\varepsilon(t)}{t}\,dt=\ln a-\alpha+\sum_{k=1}^{2n}(-1)^{k-1}\alpha_k\to\ln a-\alpha+\alpha=\ln a\,(n\to\infty),$$

$$(7)\quad \int_1^{x_{2n+2}} \frac{\varepsilon(t)}{t}\,dt = \ln a-\alpha+\rho+\sum_{k=1}^{2n+1}(-1)^{k-1}\alpha_k\to\ln a-\alpha+\rho+\alpha=\ln b\,(n\to\infty).$$

By (1) and (5), we have, for each $n\in N$,

$$\int_1^{x_{2n-1}} \frac{\varepsilon(t)}{t}dt<\int_1^{x} \frac{\varepsilon(t)}{t}\,dt\leq\int_1^{x_{2n}} \frac{\varepsilon(t)}{t}\,dt\quad (x_{2n-1}<x\leq x_{2n}),$$

$$\int_1^{x_{2n+1}} \frac{\varepsilon(t)}{t}\,dt\leq\int_1^{x} \frac{\varepsilon(t)}{t}\,dt<\int_1^{x_{2n}} \frac{\varepsilon(t)}{t}\,dt\quad (x_{2n}<x\leq x_{2n+1}),$$

wherefrom we obtain

$$\int_1^{y_x} \frac{\varepsilon(t)}{t}\,dt\leq\int_1^{x} \frac{\varepsilon(t)}{t}\,dt\leq\int_1^{z_x} \frac{\varepsilon(t)}{t}\,dt\quad (x>x_1).$$

where $x\in(x_k,x_{k+1}]$, $\{y_x,z_x\}=\{x_k,x_{k+1}\}$, $y_x\in\{x_{2\nu-1}:\nu\in N\}$, $z_x\in\{x_{2\nu}:\nu\in N\}$. This implies, according to (6) and (7),

$$\ln a\leq \varliminf_{x\to+\infty}\int_1^{x} \frac{\varepsilon(t)}{t}\,dt\leq \varlimsup_{x\to+\infty}\int_1^{x} \frac{\varepsilon(t)}{t}\,dt\leq\ln b,$$

and this together with (6) and (7) implies

$$\underline{\lim}_{x\to+\infty} \int_1^x \frac{\varepsilon(t)}{t}\,dt = \ln a, \quad \overline{\lim}_{x\to+\infty} \int_1^x \frac{\varepsilon(t)}{t}\,dt = \ln b.$$

Therefore,

$$\underline{\lim}_{x\to+\infty} L(x)=a, \quad \overline{\lim}_{x\to+\infty} L(x)=b.$$

In the case

$$0<a<b=+\infty,$$

let us pose

$$x_1=3;x_{2n}=e^{e^n-1}(x_{2n-1}+1)^{e^n}+1(n\in N),x_{2n+1}=e^{e^n-1}(x_{2n}+1)^{e^n}+1(n\in N).$$

Then, we have

$$x_{2n}>1+e^{e-1}x_{2n-1}>1+2x_{2n-1}(>x_{2n-1}) \ (n\in N),$$

$$x_{2n+1}>1+2x_{xn}(>x_{2n}) \ (n\in N),$$

that is, $x_{n+1}>1+2x_n(>x_n)$ $(n\in N)$, and so $x_{n+1}-x_n>1+x_n\geq 1+3>3$.

Therefore, in this case we shall also define $\eta(x)$ for $x\geq 2$ by (5) with $\delta=1$, and for $x\in[1,2]$ as a continuous function such that

$$\int_1^{x_1} \frac{\eta(t)}{t(1+\ln t)}\,dt=\int_1^{x_1} \frac{\varepsilon(t)}{t}\,dt = \ln a-\alpha,$$

where

$$\alpha \overset{\text{def}}{=} \sum_{k=1}^{\infty}(-1)^{k-1}\alpha_k \ ,$$

with

$$\alpha_k=\left(\int_{x_k}^{x_k+1}+\int_{x_{k+1}-1}^{x_{k+1}}\right)\frac{|n(t)|}{t(1+\ln t)}\,dt \qquad (k\in N),$$

is a finite number. Similarly as in the first case, one can establish that now

$$\int_1^{x_{2n+1}}\frac{\varepsilon(t)}{t}\,dt = \ln a-\alpha+\sum_{k=1}^{2n}(-1)^{k-1}\alpha_k\to\ln a-\alpha+\alpha=\ln a \quad (n\to\infty),$$

$$\int_1^{x_{2n+2}}\frac{\varepsilon(t)}{t}\,dt=\ln a-\alpha+n+1+\sum_{k=1}^{2n+1}(-1)^{k-1}\alpha_k\to+\infty(n\to\infty),$$

wherefrom one concludes that

$$\underline{\lim}_{x\to+\infty}\int_1^x\frac{\varepsilon(t)}{t}\,dt=\ln a, \qquad \overline{\lim}_{x\to+\infty}\int_1^x\frac{\varepsilon(t)}{t}\,dt=+\infty,$$

i.e. $\underline{\lim}_{x\to+\infty} L(x)=a$, $\overline{\lim}_{x\to+\infty} L(x)=+\infty$.

In the case

$a=0<b<+\infty$,

it is sufficient to observe that the function $L(x)=1/L_0(x)$, where L_0 is a slowly varying function continuous on $[0,+\infty)$ and such that $\underline{\lim}_{x\to+\infty} L_0(x)=1/b$, $\overline{\lim}_{x\to+\infty} L_0(x)=+\infty$ (preceding case) - is continuous on $[0,+\infty)$ and has the properties $\underline{\lim}_{x\to+\infty} L(x)=0$, $\overline{\lim}_{x\to+\infty} L(x)=b$.

Let us finally consider the case

$a=0$, $b=+\infty$.

Putting

$$x_1=3;\ x_{n+1}=1+e^{e^n-1}(x_n+1)^{e^n} \quad (n\in N),$$

and defining η(x) on [2,+∞) by (5) with δ=1, and on [1,2] as a continuous function on that interval, we get in this case

$$\int_1^{x_{n+1}} \frac{\varepsilon(t)}{t}\,dt = \int_1^{x_n} \frac{\varepsilon(t)}{t}\,dt + \sum_{x=1}^{n}(-1)^{k-1}k + \sum_{k=1}^{n}(-1)^{k-1}\alpha_k,$$

where α_k has the same meaning as in the proceding case, and consequently

$$\int_1^{x_{2n+1}} \frac{\varepsilon(t)}{t}\,dt = \int_1^{x_1} \frac{\varepsilon(t)}{t}\,dt - n + \sum_{k=1}^{2n}(-1)^{k-1}\alpha_k \to -\infty \quad (n\to\infty),$$

$$\int_1^{x_{2n+2}} \frac{\varepsilon(t)}{t}\,dt = \int_1^{x_1} \frac{\varepsilon(t)}{t}\,dt + n + 1 + \sum_{k=1}^{2n+1}(-1)^{k-1}\alpha_k \to +\infty (n\to\infty).$$

Hence,

$$\underline{\lim}_{x\to+\infty} \int_1^{x} \frac{\varepsilon(t)}{t}\,dt = -\infty, \qquad \overline{\lim}_{x\to+\infty} \int_1^{x} \frac{\varepsilon(t)}{t}\,dt = +\infty,$$

that is,

$$\underline{\lim}_{x\to+\infty} L(x)=0, \qquad \overline{\lim}_{x\infty+\infty} L(x)=+\infty.$$

<u>Remark</u>. A particular example of a slowly varying function L(x) with properties $\underline{\lim}_{x\to+\infty} L(x)=0$ and $\overline{\lim}_{x\to+\infty} L(x)=+\infty$ is given in [2] (p.58,ex.3), without detailed proof. Our result is evidently more general.

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REZIME

JEDNA PRIMEDBA KOJA SE ODNOSI NA SPORO PROMENLJIVE U SMISLU KARAMATE

Dat je kompletan dokaz jednog rezultata iz rada [1].

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