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ON TWO DEFINITIONS OF THE NON-COMMUTATIVE CONVOLUTION PRODUCT OF DISTRIBUTIONS

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ABSTRACT

Two generalizations of the standard definition of the convolution product of two generalized functions are given. It is shown that neither is a generalization of the other.

The standard way of defining the convolution product $f * g$ of certain pairs of distributions f and g is as follows, see for example Gel'fand and Shilov [7].

DEFINITION 1. Let f and g be distributions satisfying either of the following conditions:

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Then the convolution product $f * g$ is defined by

$$\langle (f * g)(x), \phi(x) \rangle = \langle g(y), \langle f(x), \phi(x + y) \rangle \rangle$$

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for arbitrary test function ϕ in the space $\mathcal{D}$ of infinitely differentiable functions with compact support.

This definition of the convolution product is very restrictive and in order that further convolution products could be defined, the next definition was introduced in [2].

DEFINITION 2. Let f and g be distributions and let τ be an infinitely differentiable function satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \leq \tau(x) \leq 1$,

(iii) $\tau(x) = 1, \qquad |x| \leq 1/2$,

(iv) $\tau(x) = 0, \qquad |x| \geq 1$.

Let $f_n(x) = f(x)\tau(x/n)$ for $n = 1, 2, \ldots$. Then the convolution product $f * g$ is defined as the limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\lim_{n\to\infty} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle$$

for all ϕ in $\mathcal{D}$.

In this definition the convolution product $f_n * g$ is in the sense of Definition 1, the distribution f_n having bounded support.

It was proved in [2] that if a convolution product $f * g$ existed by Definition 1 then it also existed by Definition 2 and defined the same distribution. Definition 2 is therefore a generalization of Definition 1. Although the convolution product of Definition 1 is always commutative, the convolution product of Definition 2 does give some convolution products which are non-commutative.

The next definition also generalizes Definition 1 and was given in [3].

DEFINITION 3. Let f and g be distributions and let τ_n be the infinitely differentiable function defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

where τ is defined as in Definition 2. Let $f_n(x) = f(x)\tau_n(x)$ for $n = 1, 2, \ldots$. Then the convolution product $f * g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that

$$\underset{n\to\infty}{\mathrm{N-lim}}\langle f_n * g, \phi\rangle = \langle h, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \qquad \ln^r n$$

for $\lambda > 0$ and $r = 1, 2, \ldots$, and all functions $\epsilon(n)$ for which $\lim_{n\to\infty} \epsilon(n) = 0$.

The convolution product $f_n * g$ in this definition is again in the sense of Definition 1, the support of f_n being contained in the interval $[-n - n^{-n}, n + n^{-n}]$.

It was shown in [3] that all the convolution products which were proved to exist by Definition 2 in [2] also existed by Definition 3. Examples were also given where the convolution product existed by Definition 3 but did not exist by Definition 2. Since then, in [4], [5] and [6], further examples were given where the convolution product existed by Definition 3 but not by Definition 2. All this suggests that every convolution product which exists by Definition 2 must also exist by Definition 3. This however is not the case and we now give an example where a convolution product exists by Definition 2 but not by Definition 3.

EXAMPLE. The convolution product

$$\sum_{k=1}^{\infty} 2^{-k}\delta'(x - k - \alpha k^{-k}) * \sum_{k=1}^{\infty} \delta(x + k + \alpha k^{-k}),$$

where $1/2 < \alpha < 1$, exists by Definition 2 but not by Definition 3.

PROOF. Put

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} 2^{-k}\delta'(x - k - \alpha k^{-k}), \\ g(x) &= \sum_{k=1}^{\infty} \delta(x + k + \alpha k^{-k}). \end{aligned}$$

Then with $n \geq 3$,

$$f_n(x) = \sum_{k=1}^{\infty} 2^{-k}\delta'(x - k - \alpha k^{-k})\tau(x/n)$$

$$
\begin{aligned}
&= \sum_{k=1}^{[n/2]} 2^{-k}\delta'(x-k-\alpha k^{-k}) \\
&\qquad + \sum_{k=[n/2]+1}^{n-1} 2^{-k}\tau\left(\frac{k+\alpha k^{-k}}{n}\right)\delta'(x-k-\alpha k^{-k}) \\
&\qquad - \sum_{k=[n/2]+1}^{n-1} 2^{-k}n^{-1}\tau'\left(\frac{k+\alpha k^{-k}}{n}\right)\delta(x-k-\alpha k^{-k}) \\
&= f_{1n}(x)+f_{2n}(x)-f_{3n}(x),
\end{aligned}
$$

where $[n/2]$ denotes the integer part of $n/2$.

The convolution products $f_{jn} * g, j = 1,2,3$, exist by Definition 1 and

$$
\begin{aligned}
f_{1n} * g &= \sum_{k=1}^{\infty}\sum_{i=1}^{[n/2]} 2^{-i}\delta'(x+k-i+\alpha k^{-k}-\alpha i^{-i}), \\
f_{2n} * g &= \sum_{k=1}^{\infty}\sum_{i=[n/2]+1}^{n-1} 2^{-i}\tau\left(\frac{i+\alpha i^{-i}}{n}\right)\delta'(x+k-i+\alpha k^{-k}-\alpha i^{-i}), \\
f_{3n} * g &= \sum_{k=1}^{\infty}\sum_{i=[n/2]+1}^{n-1} n^{-1}2^{-i}\tau'\left(\frac{i+\alpha i^{-i}}{n}\right)\delta(x+k-i+\alpha k^{-k}-\alpha i^{-i}).
\end{aligned}
$$

It is clear that the supports of these convolution products are contained in the unions of the open intervals $(r-\frac{1}{2}, r+\frac{1}{2})$ for $r = 0, \pm1, \pm2, \dots$.

For a given r let ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $(r-\frac{1}{2}, r+\frac{1}{2})$. If $r \geq 0$, then

$$
\langle f_{1n} * g, \phi\rangle = -\sum_{k=r+1}^{r+[n/2]} 2^{r-k}\phi'(r-\alpha k^{-k}+\alpha(k-r)^{r-k}),
$$

and so

$$
\lim_{n\to\infty}\langle f_{1n} * g, \phi\rangle = -\sum_{k=r+1}^{\infty} 2^{r-k}\phi'(r-\alpha k^{-k}+\alpha(k-r)^{r-k}).
$$

If $r < 0$, then

$$
\langle f_{1n} * g, \phi\rangle = -\sum_{k=1}^{r+[n/2]} 2^{r-k}\phi'(r-\alpha k^{-k}+\alpha(k-r)^{r-k}),
$$

and so

$$
\lim_{n\to\infty}\langle f_{1n} * g, \phi\rangle = -\sum_{k=1}^{\infty} 2^{r-k}\phi'(r-\alpha k^{-k}+\alpha(k-r)^{r-k}).
$$

Next, if $r \geq -[n/2]$, then

$$
\langle f_{2n} * g, \phi\rangle = -\sum_{k=r+1+[n/2]}^{n+r-1} 2^{r-k}\tau\left(\frac{k-r+\alpha(k-r)^{r-k}}{n}\right)\phi'(r-\alpha k^{-k}-\alpha(k-r)^{r-k}).
$$

Thus

$$|\langle f_{2n} * g, \phi\rangle| \leq M \sum_{k=r+1+[n/2]}^{n+r-1} 2^{r-k},$$

where

$$M = \sup\{|\phi(x)| : r - 1/2 < x < r + 1/2\}$$

and so

$$\lim_{n\to\infty} \langle f_{2n} * g, \phi\rangle = 0.$$

If $r < -[n/2]$, it follows similarly that

$$\lim_{n\to\infty} \langle f_{2n} * g, \phi\rangle = 0.$$

It can be proved similarly that

$$\lim_{n\to\infty} \langle f_{3n} * g, \phi\rangle = 0$$

for all r.

We have therefore proved that $\lim_{n\to\infty}\langle f_n * g, \phi\rangle$ exists for all test functions ϕ with supports contained in the interval $(r - \frac{1}{2}, r + \frac{1}{2})$. It now follows that the limit exists for all ϕ in $\mathcal{D}$, proving that the convolution product $f * g$ exists by Definition 2.

Now put

$$\begin{aligned} f_n(x) &= \sum_{k=1}^{\infty} 2^{-k}\delta'(x - k - \alpha k^{-k})\tau_n(x) \\ &= \sum_{k=1}^{n-1} 2^{-k}\delta'(x - k - \alpha k^{-k}) + 2^{-n}\tau_n(n + \alpha n^{-n})\delta'(x - n - \alpha n^{-n}) \\ &\quad -2^{-n}n^n\tau_n'(n + \alpha n^{-n})\delta(x - n - \alpha n^{-n}). \end{aligned}$$

The convolution product $f_n * g$ exists by Definition 1 and

$$\begin{aligned} f_n * g &= \sum_{k=1}^{\infty}\sum_{i=1}^{n-1} 2^i\delta'(x + k - i + \alpha k^{-k} - \alpha i^{-i}) \\ &\quad +2^{-n}\tau(\alpha)\sum_{k=1}^{\infty}\delta'(x + k - n + \alpha k^{-k} - \alpha n^{-n}) \\ &\quad -2^{-n}n^n\tau'(\alpha)\sum_{k=1}^{\infty}\delta(x + k - n + \alpha k^{-k} - \alpha n^{-n}). \end{aligned}$$

Let ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $(-1/2, 1/2)$

and $\phi(0) \neq 0$. Then

$$2^{-n}n^n\tau'(\alpha)\langle\sum_{k=1}^{\infty}\delta(x+k-n+\alpha k^{-k}-\alpha n^{-n}),\phi\rangle = 2^{-n}n^n\tau'(\alpha)\phi(0),$$

which is not a negligible function, unless $\tau'(\alpha) = 0$ or $\phi(0) = 0$. It follows that

$$\underset{n\to\infty}{\text{N}-\lim}\langle f_n * g, \phi\rangle$$

does not exist for all ϕ in $\mathcal{D}$ and all τ. The convolution product $f * g$ therefore does not exist by Definition 3.

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REZIME

O DVE DEFINICIJE NEKOMUTATIVNOG
KONVOLUTIVNOG PROIZVODA DISTRIBUCIJA

Data su dva uopštenja standardne definicije konvolutivnog proizvoda dve uopštene funkcije. Pokazano je da nijedno nije uopštenje drugog.

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