Univ. u Novom Sadu,
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 19,2, 1-20 (1989)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

p(3,-1) - FINSLER STRUCTURES AND THEIR LIFTS

Atanasiu Gheorghe* and Klepp C. Francisc**

**Department of Mathematics, University of Brasov, 2200 - Brasov, Romania*

***Department of Mathematics, Polytechnical Institute "Traian Vuia"*
19000 - Timisoara, Romania

Abstract

On a differentiable manifold M and on the total space E of a vector bundle there are defined $p(3,-1)$, $P(3,-1)$ - Finsler structures respectively and the Finsler connections which are compatible with these structures are determined. Furthermore the invariants of the Finsler connection transformations group are established, the integrability conditions are determined and some special cases are given.

0. Introduction

The main purpose of the present paper is to introduce the notion of $p(3,-1)$-Finsler structure on an n-dimensional C^∞- manifold and the notion of $P(3,-1)$-structure on the total space E of a vector bundle $\xi = (E,\pi,M)$ and to study these structures by the method used by R. Miron [14] and by the first author in [1].

A $P(3,-1)$-structure on E we simply call a *Walker structure.* It is similar to the Kentaro Yano structure on E, [1].

In § 1 we recall the notion of a Finsler connection on M [11], [12] and the notion of a distinguished connection on E [15], [16], [17]. We introduce, in § 2, the notion of $p(3,-1)$-Finsler structure on M and all $p(3,-1)$-Finsler connections are determined in § 3, using Obata's operators (2.5). In § 4, we consider the group G_p of transformations of $p(3,-1)$-Finsler connections and prove that it has twenty invariants, which

---

AMS Mathematics Subject Classification: 53B40
Key words and phrases: Finsler connections, p(3,-1)-Finsler connections, horizontal and vertical distribution of T(TM)

are Finsler tensor fields. By the lift in Miron's sense [14], on TM of one $p(3,-1)$-Finsler structure from M, in §5, we obtain three structures on TM, that each of them, taken adequately on the horizontal distribution HTM and on the vertical distribution VTM of $T(TM)$, are $p(3,-1)$-Finsler structures. By means of the invariants of § 3, in §6, we give characterizations of the integrability of type I, II or III for the $p(3,-1)$-Finsler structures based only on Finsler geometry. Finally, in §6, we define and treat the (h,v) $P(3,-1)$-structure on the total space E of a vector bundle ξ, which we shall call a *(h,v)-Walker structure* on E.

In particular, we have on M an almost product Finsler structure [3], [8], [9], [10], or a (p,ξ,η)-Finsler structure [6].

The terminology and notations are all according to M. Matsumoto [11] and R. Miron [12,13,14,15,16,17].

1. Preliminaries

Let M be a differentiable C^∞ - manifold, paracompact, of dimension n, let $T(M) = (TM,\pi,M)$ be its tangent bundle and let N be a non-linear connection on TM: $TM = N + (TM)^V$. We denote by $(x^i, y^i)(i,j,k,l=1,2,\dots,n)$ the canonical coordinates on TM. Then $\delta_i = \partial_i - N^k_i\dot\partial_k$ $(\partial_i = \partial/\partial x^i,\ \dot\partial_i = \partial/\partial y^i,\ \delta_i = \delta/\delta x^i)$ is a local basis of the horizontal distribution N, and $\dot\partial_i$ is a local basis of the vertical distribution $(TM)^V$. The dual basis is $(dx^i, \delta y^i)$, where $\delta y^i = dy^i + N^i_k dx^k$. We have:

$$(1.1)\qquad [\delta_j,\delta_k] = R^i_{jk}\dot\partial_i,\quad [\delta_j,\dot\partial_k] - [\delta_k,\dot\partial_j] = t^i_{jk}\dot\partial_i,\quad [\dot\partial_j,\dot\partial_k] = 0$$

We denote by

$$(1.2)\qquad R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k,\qquad t^i_{jk} = \dot\partial_k N^i_j - \dot\partial_j N^i_k$$

the curvature and torsion tensor fields of N.

A Finsler connection on M is a triad $FT = (N,F,C)(=\nabla)$, where N is a non-linear connection on TM, and F, respectively, C are the h-and v-connection coefficients given by:

$$(1.3)\qquad \nabla_{\delta_k}\delta_j = F^i_{jk}\delta_i,\ \nabla_{\delta_k}\dot\partial_j = F^i_{jk}\dot\partial_i,\ \nabla_{\dot\partial_k}\delta_j = C^i_{jk}\delta_i,\ \nabla_{\dot\partial_k}\dot\partial_j = C^i_{jk}\dot\partial_i$$

With $|$ and $\Big|$ we denote the h-and v-covariant derivatives with respect to FT. For example, for a Finsler tensor field of the type (1,1) with the components $K^i_j(x,y)$ we have

$$(1.4)\qquad K^i_{j|k} = \delta_k K^i_j + F^i_{sk}K^s_j - F^s_{jk}K^i_s,\quad K^i_j\Big|_k = \dot\partial_k K^i_j + C^i_{sk}K^s_j - C^s_{jk}K^i_s .$$

The torsion Finsler tensor filed of $F\Gamma$ will be denoted by: T^i_{jk}, R^i_{jk}, C^i_{jk}, P^i_{jk}, S^i_{jk}, and the curvature Finsler tensor fields of $F\Gamma$ will be denoted by: $R_j{}^i{}_{kh}$, $P_j{}^i{}_{kh}$, $S_j{}^i{}_{kh}$, [11], [12].

If $F\Gamma = (N,F,C)$ and $F\bar\Gamma = (\bar N,\bar F,\bar C)$ are two Finsler connections on M, then a unique triad of Finsler tensor fileds (A,B,D) is determined such that:

$$(1.5)\qquad \bar N^i_j = N^i_j - A^i_j;\quad \bar F^i_{jk} = F^i_{jk} + C^i_{js}A^s_k - B^i_{jk},\quad \bar C^i_{jk} = C^i_{jk} - D^i_{jk} .$$

Conversely: given the Finsler connection $F\Gamma = (N,F,C)$ and the triad (A,B,D) of Finsler tensor fields, the connection $F\bar\Gamma = (\bar N,\bar F,\bar C)$ given by (1.5) is a Finsler connection. The map $F\Gamma \to F\bar\Gamma$ defined by (1.5) is called a transformation of Finsler connections.

If $F\mathring\Gamma = (\mathring N,\mathring F,\mathring C)$ is a connection fixed on M, then the h - and v - covariant derivatives with respect to $F\mathring\Gamma$ will be denoted, resceptively, by $\mathring{|}$ and $\mathring{\Big|}$.

Because a linear connection $D = (\Gamma^i_{jk})$ on TM, expressed in the adapted basis $(\delta_i,\dot\partial_i)$ of a non-linear connection N and the vertical distribution $(TM)^V$ has eight coefficients, in [15], [16] was introduced the notion of a d-connection on the total space E of a vector bundle $\xi = (E,\pi,M)$ (in particularly on $E = TM$), that is, fundametal in the geometry of E. We shall recall in short this important notion.

Let $\xi = (E,\pi,M)$ be a vector bundle of the class C^∞. We suppose that the total space E has $n+m$ dimensions, the base M has n dimensions and the local fibre $E_x = \pi^{-1}(x)$, $x \in M$, is a real vector space of dimensions m. $\mathcal{X}(E)$, $\mathcal{X}^*(E)$, $\sigma^p_q(E)$ are well-known sections.

We denote by N a non-linear connection (horizontal distribution) on E and by V the vertical distribution complementary with N:

$$T_uE = H_uE + V_uE,\quad \forall u \in E.$$

A linear connection D on E is called a *distinguished connection* (for short a d-connection) if: $D_ZX \in HE$, $D_ZY \in VE$, for $\forall X \in HE$, $\forall Y \in VE$, $\forall Z \in \mathcal{X}(E)$.

It follows that for a d-connection on E we have a unique decomposition: $D = D^H + D^V$. D^H is the h-covariant derivatives and D^V is v-covariant derivative of D.

In the canonical coordinates (x^i, y^a) of the point $u = (u^\alpha) \in E$, $i=\overline{1,n}$, $a=\overline{1,m}$, $\alpha=\overline{1,n+m}$, we have $(\delta_i, \dot\partial_a)$, $(dx^i, \delta y^a)$ the dual frames adapted to N $(N^a_i(x,y))$.

The local components $(L^i_{jk}(x,y),\ L^a_{bk}(x,y),\ C^i_{jc}(x,y),\ C^a_{bc}(x,y))$ of a d-connection on E are given by:

$$(1.6)\qquad D_{\delta_k}\delta_j = L^i_{jk}\delta_i,\quad D_{\delta_k}\dot\partial_b = L^a_{bk}\dot\partial_a,\quad D_{\dot\partial_c}\delta_j = C^i_{jc}\delta_i,\quad D_{\dot\partial_c}\dot\partial_b = C^a_{bc}\dot\partial_a .$$

We denote by $X^H(X^V)$ and $\omega^H(\omega^V)$, the horizontal (vertical) components of $X \in \mathfrak{X}(E)$ and respectively $\omega \in \mathfrak{X}^*(E)$.

A tensor field t on E is called a *distinguished tensor field* (*d-tensor filed*) of the type $\begin{pmatrix} pr \\ qs \end{pmatrix}$ if it has the property:

$$(1.7)\qquad t\left(\underset{1}{\omega},\dots,\underset{p}{\omega};\ \underset{1}{X},\dots,\underset{q}{X};\ \underset{p+1}{\omega},\dots,\underset{p+r}{\omega};\ \underset{q+1}{X},\dots,\underset{q+s}{X}\right) =$$

$$= t\left(\underset{1}{\omega}^H,\dots,\underset{p}{\omega}^H;\ \underset{1}{X}^H,\dots,\underset{q}{X}^H;\ \underset{p+1}{\omega}^V,\dots,\underset{p+r}{\omega}^V;\ \underset{q+1}{X}^V,\dots,\underset{q+s}{X}^V\right),$$

$$\forall \underset{\alpha}{X} \in \mathfrak{X}(E),\ \forall \underset{\alpha}{\omega} \in \mathfrak{X}^*(E).$$

Proposition 1.1. *If t is a tensor field on E of the type (p,q), then it determines 2^{p+q} d-tensor flleds on E of the type* $\begin{pmatrix} p-r & r \\ q-s & s \end{pmatrix}$ $(r=0,1,\dots,p;$ $s=0,1,\dots q)$.

It follows that the torsion tensor field T of the d-connection $D\Gamma$, (1.6), is characterized by five d-tensor fields of components: T^i_{jk}, R^a_{jk}, C^i_{jc}, P^a_{jc}, S^a_{bc} $(T(X,Y) = -T(Y,X))$, where:

$$(1.8)\qquad \begin{cases} T^i_{jk} = L^i_{jk} - L^i_{kj},\quad R^a_{jk} = \delta_k N^a_j - \delta_j N^a_k,\quad C^i_{jc}, \\ P^a_{jc} = \dot\partial_c N^a_j - L^a_{cj},\quad S^a_{bc} = C^a_{bc} - C^a_{cb} . \end{cases}$$

Also, the curvature tensor filed R of the d-connection $D\Gamma$, (1.6), is characterized by six d-tensor fields of components: $R_j{}^i{}_{kl}$, $R_b{}^a{}_{kl}$, $P_j{}^i{}_{kl}$, $P_b{}^a{}_{kd}$, $S_j{}^i{}_{cd}$, $S_b{}^a{}_{cd}$ $((R(X,Y)Z^H)^V = 0,\ \ (R(X,Y)Z^V)^H = 0)$, where:

$$(1.9)_1 \quad \begin{cases} R_j{}^i{}_{kl} = \delta_l L^i_{jk} - \delta_k L^i_{jl} + L^h_{jk} L^i_{hl} - L^h_{jl} L^i_{hk} + C^i_{jc} R^c_{kl}\,, \\ R_b{}^a{}_{kl} = \delta_l L^a_{bk} - \delta_k L^a_{bl} + L^f_{bk} L^a_{fl} - L^f_{bl} L^a_{fk} + C^a_{bc} R^c_{kl}\,. \end{cases}$$

$$(1.9)_2 \quad \begin{cases} P_j{}^i{}_{kd} = \dot{\partial}_d L^i_{jk} - C^i_{jd|k} + C^i_{jf} P^f_{kd} \\ P_b{}^a{}_{kd} = \dot{\partial}_d L^a_{bk} - C^a_{bd|k} + C^a_{bf} P^f_{kd}\,, \end{cases}$$

$$(1.9)_3 \quad \begin{cases} S_j{}^i{}_{cd} = \dot{\partial}_d C^i_{jc} - \dot{\partial}_c C^i_{jd} + C^h_{jc} C^i_{hd} - C^h_{jd} C^i_{hc}\,, \\ S_b{}^a{}_{cd} = \dot{\partial}_d C^a_{bc} - \dot{\partial}_c C^a_{bd} + C^f_{bc} C^a_{fd} - C^f_{bd} C^a_{fc}\,. \end{cases}$$

If $E = TM$, we have $n=m$ and we can remain in the present notations on E, with the convention to assimilate x^a with $\delta^a_i x^i$.

Then, on TM have in general:

$$(1.10) \qquad L^i_{jk} \neq \delta^i_a\, \delta^b_j\, L^a_{bk}\,, \qquad C^i_{jc} \neq \delta^i_a\, \delta^b_j\, C^a_{bc}. \qquad ;$$

A d-connection $D\Gamma$ on TM for which (1.10) is tranformed into equalities is called a *d-connection of the Finsler type* on TM (in [23] such a d-connection is called a d-normal connection).

Remark 1.1. A d-tensor filed on the manifold base M is also called a Finsler tensor filed on M [11], [12].

2. $p(3,-1)$**-Finsler structures**

Let M be an n-dimensional differentiable manifold of class C^∞ and $x=(x^i)$ and $y=(y^i)$ denote a point of M and a supporting element, respectively.

Definition 2.1. *A Finsler tensor field* $p(x,y) \neq 0$ *of type* (1,1) *and of class* C^∞ *is called a* $p(3,-1)$*-Finsler structures of index* k, *it is satisfies*

$$(2.1) \qquad p^3 - p = 0, \quad rank\, \| p(x,y) \| = n - k\,, \quad \forall (x,y) \in TM\,,$$

where k *is integer and* $0 \le k < n$.

We denote by $\mathcal{H}$ and V complementary distributions of tangent bundle $T(M)$, corresponding to the projection operators $h(x,y)$ and $v(x,y)$ given by

(2.2) $$h = p^2, \quad v = -p^2 + I \qquad (I \text{ - the identity operator}).$$

We have $\dim \mathcal{H}_{(x,y)} = n - k$, $\dim V_{(x,y)} = k$ and

(2.3) $$\begin{cases} ph = hp = p & pv = vp = o \\ p^2 h = h & p^2 v = o \end{cases}$$

that is, p acts on $\mathcal{H}$ as an almost product operator and on V as a null operator.

Remark 2.1. If the rank of $p(x,y)$ is n, then $h = I$ and $v = 0$ and $p(x,y)$ satisfies: $p^2 = I$. Consequently the $p(3,-1)$-Finsler structure of ***minimum index*** is an almost product Finsler structure [3], [8], [9]; the dimension n not must be even.

Remark 2.2. If the $p(3,-1)$-Finsler structure is of *index 1*, then $\mathcal{H}$ is $(n-1)$-dimensional and V is one-dimensional. Consequently if we denote by $p^i_j(x,y)$, $h^i_j(x,y)$ and $v^i_j(x,y)$ $(i,j,\ldots,= 1,2,\ldots,n)$ the local components of $p(x,y)$, $h(x,y)$ and $v(x,y)$ respecitvely, then $v^i_j(x,y)$ should have the form $v^i_j = \eta_j \xi^i$, where $\eta(x,y)$ and $\xi(x,y)$ are covariant contravariant Finsler vector fields respectively. Taking into account the relations (2.2) and (2.3) we have

(2.4) $$p^i_r p^r_j = \delta^i_j - \eta_j \xi^i, \quad p^i_j \xi^j = o, \quad \eta_i p^i_j = o, \quad \eta_i \xi^i = 1 .$$

Thus a $p(3,-1)$-Finsler structure of index 1 is equivalent to a (p,ξ,η)-Finsler structure (see [6], p.26, Definition 3.2 and Remark 3.2).

Definition 2.2. *We shall call Obata's operators of the $p(3,-1)$-Finsler structure,* (2.1), *the Finsler tensor fields* $\underset{1}{O}{}^{rs}_{ij}$ *and* $\underset{2}{O}{}^{rs}_{ij}$ *given by*

(2.5) $$\begin{cases} \underset{1}{O}{}^{rs}_{ij} = \frac{1}{2}\left[\delta^r_i \delta^s_j - \delta^r_i v^s_j - v^r_i \delta^s_j + p^r_i p^s_j + 3\, v^r_i v^s_j \right], \\ \underset{2}{O}{}^{rs}_{ij} = \frac{1}{2}\left[\delta^r_i \delta^s_j + \delta^r_i v^s_j + v^r_i \delta^s_j - p^r_i p^s_j - 3\, v^r_i v^s_j \right], \end{cases}$$

These operators have the symmetry $\underset{\alpha}{O}{}^{rs}_{ij} = \underset{\alpha}{O}{}^{sr}_{ji}$ (α=1,2) and act on a Finsler tensor field K of type (1,2) as $(\underset{\alpha}{O}K)^{i}_{jk} = \underset{\alpha}{O}{}^{ri}_{js} K^{s}_{rk}$. The product $\underset{\alpha}{O}\,\underset{\beta}{O}$ is defined by $(\underset{\alpha}{O}\,\underset{\beta}{O})^{ti}_{jm} = \underset{\alpha}{O}{}^{ri}_{js}\,\underset{\beta}{O}{}^{ts}_{rm}$ (α,β = 1,2).

Proposition 2.1. $\underset{1}{O},\underset{2}{O}$ *are the supplementary projectors on the module* τ^{1}_{2} *of the tensor fields of type* (1,2):

$$(2.6)\qquad \underset{1}{O} + \underset{2}{O} = I, \quad \underset{\alpha}{O}{}^{2} = \underset{\alpha}{O}\,, \quad \underset{1}{O}\,\underset{2}{O} = \underset{2}{O}\,\underset{1}{O} = o \quad (\alpha=1,2)\,,$$

where I *is the identity given by* $\delta^{r}_{j}\,\delta^{i}_{s}$: $IK = K$.

Proposition 2.2. $\underset{2}{O}X = o$ *(resp.* $\underset{1}{O}X = o$*) has solutions, and its general solutions are given by* $X = \underset{1}{O}Y$ *(resp.* $X = \underset{2}{O}Y$*), where* $Y \in \tau^{1}_{2}$ *is arbitrary.*

3. $p(3,-1)$-Finsler connections

An important problem concerning a $p(3,1)$-Finsler connections on M is to determine the existence and arbitrariness of Finsler connections with respect to wich $p^{i}_{j}(x,y)$ is covariantly constant.

Definition 3.1. *Let* $p^{i}_{j}(x,y)$ *be a* $p(3,1)$*-Finsler structure of index k. A Finsler connection* $F\Gamma = (N,F,C)$ *is called a* $p(3,1)$*-Finsler connection or compatible with* p^{i}_{j}(2.1), *if* p^{i}_{j} *is covariantly constant:*

$$(3.1)\qquad p^{i}_{j|k} = 0 \qquad p^{i}_{j}\Big|_{k} = 0$$

Proposition 3.1. *With respect to a Finsler connection* $F\Gamma$ *compatible with a* $p(3,1)$*-Finsler structure* $p^{i}_{j}(x,y)$, (2.1), *the tensor fields* $h^{i}_{j}(x,y)$ and $v^{i}_{j}(x,y)$ *are covariantly cosntant:*

$$(3.2)\qquad h^{i}_{j|k} = o, \quad h^{i}_{j}\Big|_{k} = o, \quad v^{i}_{j|k} = o, \quad v^{i}_{j}\Big|_{k} = o\,.$$

Theorem 3.1. *(a) Obata tensor fields* $\underset{1}{O}$ *and* $\underset{2}{O}$ *are covariantly constant with respect to any* $p(3,1)$*-Finsler connections* $F\Gamma$*; (b) The Finsler tensor fields* $\underset{2}{O}{}^{ir}_{sj}R_{r}{}^{s}{}_{kl}$, $\underset{2}{O}{}^{ir}_{sj}P_{r}{}^{s}{}_{kl}$, $\underset{2}{O}{}^{ir}_{sj}S_{r}{}^{s}{}_{kl}$ and *h-and v-covariant derivatives of every order vanish for every* $F\Gamma$ *with the property* (3.1).

Proof. The property (a) results immediately from (2.5), (3.1) and (3.2). By the Ricci identities applying to p^i_j and taking into account (a) we get the statement (*b*).

Theorem 3.2. *Let* $\overset{\circ}{F\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ *be a fixed Finsler connection on M. Then there exist p(3,1)-Finsler connections with respect to the p(3,-1)-Finsler structures* (2.1); *one of these is*:

$$(3.3)\quad \begin{cases} \overset{\vee}{N}{}^i_j = \overset{\circ}{N}{}^i_j, \quad \overset{\vee}{F}{}^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \frac{1}{2}\left\{p^i_r p^r_j \overset{\circ}{|}_k - v^i_j \overset{\circ}{|}_k + 3v^i_r v^r_j \overset{\circ}{|}_k\right\} \\ \overset{\vee}{C}{}^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \frac{1}{2}\left\{p^i_r p^r_j \overset{\circ}{|}_k - v^i_j \overset{\circ}{|}_k + 3v^i_r v^r_j \overset{\circ}{|}_k\right\} \end{cases}$$

Proof. By straightforward calculus, (3.3) satisfies (3.1).

Now, we determine all the *p*(3,-1)-Finsler connections based on Proposition 2.2. Let $\overset{\bullet}{F\Gamma}$ be a fixed Finsler connection. Then any Finsler connection $F\Gamma$ on *M* can be expressed in the form

$$(3.4)\quad N^i_j = \overset{\bullet}{N}{}^i_j - A^i_j, \quad F^i_{jk} = \overset{\bullet}{F}{}^i_{jk} + \overset{\bullet}{C}{}^i_{jk} A^r_k - B^i_{jk}, \quad C^i_{jk} = \overset{\bullet}{C}{}^i_{jk} - D^i_{jk},$$

where *A*, *B*, *D* are arbitrary Finsler tensor fields [12], [13].

We put $\overset{\bullet}{F\Gamma} = \overset{\vee}{F\Gamma}$ in (3.4), where $\overset{\vee}{F\Gamma} = (\overset{\vee}{N}, \overset{\vee}{F}, \overset{\vee}{C})$ is given by (3.3).

In order that $F\Gamma$ is a *p*(3,-1)-Finsler connection, that is, the equation (3.1) hold for $F\Gamma$ given by (3.4), it is necessary and sufficient that *A*, *B*, *D* satisfy

$$B^i_{jk} - (p^i_s p^h_j + v^i_s v^h_j) B^s_{hk} = 0, \qquad D^i_{jk} - (p^i_s p^h_j + v^i_j v^h_j) D^s_{hk} = 0,$$

which, after one long calculus (analogous to [6], p.23-24), is equivalent to

$$\underset{2}{O}\, B = 0 \qquad \underset{2}{O}\, D = 0 .$$

From Proposition 2.2, however, the last system has solutions in B^i_{jk}, D^i_{jk} for any Finsler tensor field $A^i_j = X^i_j$. Thus, we have

Theorem 3.3. *Let* $\overset{\circ}{F\Gamma}$ *be a fixed Finsler connection. The set of all p(3,-1)-Finsler connection* $F\Gamma$ *with respect to the p(3,-1)-Finsler structure* (2.1) *is given by*

$$
(3.5)\quad \begin{cases}
N^i_j = \overset{\circ}{N}{}^i_j - X^i_j\,, \\[4pt]
F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jr} X^r_k + \dfrac{1}{2}\Big\{ p^i_s \Big[p^s_j \overset{\circ}{|}_k + p^s_j \overset{\circ}{|}_m X^m_k \Big] - \Big[v^i_j \overset{\circ}{|}_k + v^i_j \overset{\circ}{|}_m X^m_k \Big] + \\
\qquad 3 v^i_s \Big[v^s_j \overset{\circ}{|}_k + v^s_j \overset{\circ}{|}_m X^m_k \Big] \Big\} + \underset{1}{O}{}^{is}_{mj}\, Y^m_{sk}\,, \\[4pt]
C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \dfrac{1}{2}\Big\{ p^i_s p^s_j \overset{\circ}{|}_k + v^i_j \overset{\circ}{|}_k + 3 v^i_s v^s_j \overset{\circ}{|}_k \Big\} + \underset{1}{O}{}^{is}_{mj}\, Z^m_{sk}\,,
\end{cases}
$$

where X^i_j, Y^i_{jk}, Z^i_{jk} *are arbitrary Finsler tensor fields.*

Remark 3.1. The p(3,-1)-Finsler connection $F\overset{\vee}{\Gamma} = (\overset{\vee}{N}, \overset{\vee}{F}, \overset{\vee}{C})$ given by (3.3) is obtained from (3.5) for $X^i_j = 0$, $Y^i_{jk} = 0$, $Z^i_{jk} = 0$.

Corrolary 3.1. *If* $F\overset{\circ}{\Gamma}$ *is a fixed p(3,-1)-Finsler connection, then the set of all p(3,-1)-Finsler connections* $F\Gamma$ *is given by:*

$$
(3.6)\quad N^i_j = \overset{\circ}{N}{}^i_j - X^i_j\,,\quad F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \overset{\circ}{C}{}^i_{jr} X^r_k + \underset{1}{O}{}^{ir}_{sj}\, Y^s_{rk}\,,\quad C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \underset{1}{O}{}^{ir}_{sj}\, Z^s_{rk}\,,
$$

where X^i_j, Y^i_{jk}, Z^i_{jk} *are arbitrary Finsler tensor fields.*

We denote by $F\Gamma(\overset{\circ}{N})$ the Finsler connections having the same non-linear connection N.

Theorem 3.4. *The set of all p(3,-1)-Finsler connections* $F\Gamma(\overset{\circ}{N})$ *is given by*

$$
(3.7)\quad \begin{aligned}
F^i_{jk} &= \overset{\circ}{F}{}^i_{jk} + \frac{1}{2}\Big\{ p^i_s\, p^s_j \overset{\circ}{|}_k - v^i_j \overset{\circ}{|}_k + 3 v^i_s v^s_j \overset{\circ}{|}_k \Big\} + \underset{1}{O}{}^{is}_{jr}\, Y^r_{sk}\,, \\
C^i_{jk} &= C^i_{jk} + \frac{1}{2}\Big\{ p^i_s\, p^s_j \overset{\circ}{|}_k - v^i_j \overset{\circ}{|}_k + 3 v^i_s v^s_j \overset{\circ}{|}_k \Big\} + \underset{1}{O}{}^{is}_{jr}\, Z^r_{sk}\,,
\end{aligned}
$$

where $F\overset{\circ}{\Gamma}$ *is a fixed Finsler connection and* Y^i_{jk}, Z^i_{jk} *are arbitrary Finsler tensor fields.*

4. The group of transformations of p(3,-1)-Finsler connections

Let us consider the transformations $F\Gamma(N) \to F\overline{\Gamma}(N)$, [13], of p(3,-1)-Finsler connections, which preserve the non-linear connection N. Owing to Theorem 3.4. they are given by

$$
\overline{N}{}^i_j = N^i_j\,,\quad \overline{F}{}^i_{jk} = F^i_{jk} + \underset{1}{O}{}^{ir}_{sj}\, Y^s_{rk}\,,\quad \overline{C}{}^i_{jk} = C^i_{jk} + \underset{1}{O}{}^{ir}_{sj}\, Z^s_{rk}\,.
$$

Theorem 4.1. *The set of all transformations (4.1), with the mapping product as a law of composition, form an Abelian group G_p, which is isomorphic with the additive group of pair of Finsler tensor fields* $\left(\underset{1}{O}{}^{ir}_{sj}\,Y^{s}_{rk},\ \underset{1}{O}{}^{ir}_{sj}\,Z^{s}_{rk}\right)$.

By a straightforward calculus we can prove

Theorem 4.2. *The following Finsler tensor fields are invariants by the action of the group G_p:*

$$R^{i}_{jk}\ ,\quad t^{i}_{jk}\ , \tag{4.2}$$

$$\overset{1}{N}{}^{i}_{jk} = \underset{2}{O}{}^{ri}_{pq}\,\underset{1}{O}{}^{ps}_{jk}\,T^{q}_{rs}\ ,\quad \tilde{C}^{i}_{jk} = \underset{2}{O}{}^{ri}_{pq}\,\underset{1}{O}{}^{ps}_{jk}\,C^{q}_{rs}\ ,\quad \tilde{P}^{i}_{jk} = \underset{2}{O}{}^{ri}_{pq}\,\underset{1}{O}{}^{ps}_{jk}\,P^{q}_{rs}\ ,\quad \overset{2}{N}{}^{i}_{jk} = \underset{2}{O}{}^{ri}_{pq}\,\underset{1}{O}{}^{ps}_{jk}\,S^{q}_{rs}\ , \tag{4.3}$$

$$\begin{cases} \overset{1}{T}{}^{i}_{jk} = h^{i}_{m}\,T^{m}_{jk} + p^{r}_{j}\,p^{s}_{k}\,T^{i}_{rs} - \left(p^{r}_{j}\,T^{m}_{rk} + p^{s}_{k}\,T^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{1}{R}{}^{i}_{jk} = h^{i}_{m}\,R^{m}_{jk} + p^{r}_{j}\,p^{s}_{k}\,R^{i}_{rs} - \left(p^{r}_{j}\,R^{m}_{rk} + p^{s}_{k}\,R^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{1}{C}{}^{i}_{jk} = h^{i}_{m}\,C^{m}_{jk} + p^{r}_{j}\,p^{s}_{k}\,C^{i}_{rs} - \left(p^{r}_{j}\,C^{m}_{rk} + p^{s}_{k}\,C^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{1}{P}{}^{i}_{jk} = h^{i}_{m}\,P^{m}_{jk} + p^{r}_{j}\,p^{s}_{k}\,P^{i}_{rs} - \left(p^{r}_{j}\,P^{m}_{rk} + p^{s}_{k}\,P^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{1}{S}{}^{i}_{jk} = h^{i}_{m}\,S^{m}_{jk} + p^{r}_{j}\,p^{s}_{k}\,S^{i}_{rs} - \left(p^{r}_{j}\,S^{m}_{rk} + p^{s}_{k}\,S^{m}_{js}\right) p^{i}_{m}\ , \end{cases} \tag{4.4}$$

$$\begin{cases} \overset{2}{R}{}^{i}_{jk} = h^{i}_{m}\,R^{m}_{jk} + p^{r}_{j}\,p^{s}_{k}\,R^{i}_{rs} + \left(p^{r}_{j}\,R^{m}_{rk} + p^{s}_{k}\,R^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{2}{C}{}^{i}_{jk} = h^{i}_{m}\,C^{m}_{jk} - p^{r}_{j}\,p^{s}_{k}\,C^{i}_{rs} - \left(p^{r}_{j}\,C^{m}_{rk} - p^{s}_{k}\,C^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{2}{P}{}^{i}_{jk} = h^{i}_{m}\,P^{m}_{jk} - p^{r}_{j}\,p^{s}_{k}\,P^{i}_{rs} + \left(p^{r}_{j}\,P^{m}_{rk} - p^{s}_{k}\,P^{m}_{js}\right) p^{i}_{m}\ , \end{cases} \tag{4.5}$$

$$\begin{cases} \overset{3}{T}{}^{i}_{jk} = h^{i}_{m}\,T^{m}_{jk} + \left(p^{r}_{j}\,P^{m}_{kr} + p^{s}_{k}\,P^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{3}{R}{}^{i}_{jk} = h^{i}_{m}\,R^{m}_{jk} + p^{r}_{j}\,p^{s}_{k}\,S^{i}_{rs} + \left(p^{r}_{j}\,C^{m}_{kr} + p^{s}_{k}\,C^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{3}{C}{}^{i}_{jk} = h^{i}_{m}\,C^{m}_{jk} - p^{r}_{j}\,p^{s}_{k}\,C^{i}_{sr} - \left(p^{r}_{j}\,S^{m}_{rk} + p^{s}_{k}\,R^{m}_{js}\right) p^{i}_{m}\ ,\\ \overset{3}{P}{}^{i}_{jk} = h^{i}_{m}\,P^{m}_{jk} - p^{r}_{j}\,p^{s}_{k}\,P^{i}_{sr} - p^{s}_{k}\,T^{m}_{js}\,p^{i}_{m}\ , \end{cases} \tag{4.6}$$

$$(4.6)\quad \begin{cases} \overset{3}{S}{}^i_{jk} = h^i_m S^m_{jk} + p^r_j p^s_k R^i_{rs} - \left[p^r_j S^m_{rk} + p^s_k R^m_{js}\right] p^i_m , \\ \tilde{T}^i_{jk} = p^r_j p^s_k T^i_{rs} + \left[p^r_j P^m_{rk} + p^s_k P^m_{sj}\right] p^i_m , \end{cases}$$

Theorem 4.3. *The Finsler tensor fields* $\overset{1}{N}{}^i_{jk}$, $\overset{2}{N}{}^i_{jk}$, $\overset{1}{T}{}^i_{jk}$, $\overset{1}{S}{}^i_{jk}$ *vanish if and only if there exists on M an h - and v - semi-symmetric p(3,-1)-Finsler connection* $F\Gamma$ *(N).*

5. $\tilde{P}(3,-1)$-structures on the tangent bundle $T(M)$

The lift of one p(3,-1)-Finsler structure. Let M be a differentiable manifold of the class C^∞, with n dimensions, $T(M) = (TM,\pi,M)$ its tangent bundle and N a fixed non-linear connection on TM.

A $\tilde{P}(3,-1)$-structure of index k' on TM is given by a tensor field $\tilde{P} \in \tau^1_1(TM)$ with the property:

$$(5.1)\qquad \tilde{P}^3 - \tilde{P} = 0,\ \text{rank}\ \|\tilde{P}(x,y)\| = 2n - k';\quad 0 \le k' < 2n,\quad \forall (x,y) \in TM$$

In the adapted basis $X_A = \{\delta_i, \dot{\partial}_i\}$, $A = \overline{(1,2n)}$, $i = \overline{(1,n)}$, $\tilde{P}$ can be represented by:

$$(5.2)\qquad \tilde{P} = \overset{1}{P}{}^i_j\, \delta_i \otimes dx^j \otimes \overset{2}{P}{}^i_j\, \delta_i \otimes \delta y^j + \overset{3}{P}{}^i_j\, \dot{\partial}_i \otimes dx^j + \overset{4}{P}{}^i_j\, \dot{\partial}_i \otimes \delta y^j ,$$

where $\overset{\alpha}{P}{}^i_j$ $(\alpha = 1,2,3,4)$ are Finsler tensor fields on M.

Hence, we have

$$(5.3)\qquad \tilde{P}(\delta_j) = \overset{1}{P}{}^i_j\, \delta_i + \overset{3}{P}{}^i_j\, \dot{\partial}_i ,\qquad \tilde{P}(\dot{\partial}_j) = \overset{2}{P}{}^i_j\, \delta_i + \overset{4}{P}{}^i_j\, \dot{\partial}_i ,$$

and the condition (5.1) is equivalent with

$$(5.4)\quad \begin{cases} \left(\overset{1}{P}{}^i_h \overset{1}{P}{}^h_s + \overset{2}{P}{}^i_h \overset{3}{P}{}^h_s\right) \overset{1}{P}{}^s_j + \left(\overset{1}{P}{}^i_h \overset{2}{P}{}^h_s + \overset{2}{P}{}^i_h \overset{4}{P}{}^h_s\right) \overset{3}{P}{}^s_j - \overset{1}{P}{}^i_j = 0 , \\ \left(\overset{1}{P}{}^i_h \overset{1}{P}{}^h_s + \overset{2}{P}{}^i_h \overset{3}{P}{}^h_s\right) \overset{2}{P}{}^s_j + \left(\overset{1}{P}{}^i_h \overset{2}{P}{}^h_s + \overset{2}{P}{}^i_h \overset{4}{P}{}^h_s\right) \overset{4}{P}{}^s_j - \overset{2}{P}{}^i_j = 0 , \\ \left(\overset{3}{P}{}^i_h \overset{1}{P}{}^h_s + \overset{4}{P}{}^i_h \overset{3}{P}{}^h_s\right) \overset{1}{P}{}^s_j + \left(\overset{3}{P}{}^i_h \overset{2}{P}{}^h_s + \overset{4}{P}{}^i_h \overset{4}{P}{}^h_s\right) \overset{3}{P}{}^s_j - \overset{3}{P}{}^i_j = 0 , \\ \left(\overset{3}{P}{}^i_h \overset{1}{P}{}^h_s + \overset{4}{P}{}^i_h \overset{3}{P}{}^h_s\right) \overset{2}{P}{}^s_j + \left(\overset{3}{P}{}^i_h \overset{2}{P}{}^h_s + \overset{4}{P}{}^i_h \overset{4}{P}{}^h_s\right) \overset{4}{P}{}^s_j - \overset{4}{P}{}^i_j = 0 . \end{cases}$$

Also, we suppose that the components of $\tilde{P}$ fulfill conditions such as

$$H = \tilde{P}^2 , \quad V = - \tilde{P}^2 + I , \tag{5.5}$$

in order to be orthogonal projectors and supplementary.

If $p^i_j(x,y)$ is a $p(3,-1)$-Finsler structure of index k on M, then on TM, in the presence of a non-linear connection, N, we have some special cases:

$$\begin{aligned} \overset{1}{\tilde{P}} &= p^i_j \, \delta_i \otimes dx^j + p^i_j \, \dot{\partial}_i \otimes \delta y^j , \\ \overset{2}{\tilde{P}} &= p^i_j \, \delta_i \otimes dx^j - p^i_j \, \dot{\partial}_i \otimes \delta y^j , \\ \overset{3}{\tilde{P}} &= p^i_j \, \delta_i \otimes dy^j + p^i_j \, \dot{\partial}_i \otimes d x^j . \end{aligned} \tag{5.6}$$

The tensor fields $\overset{\alpha}{\tilde{P}}$ $(\alpha = 1,2,3)$ given by (5.6) are $\tilde{P}(3,-1)$-structures of a special type on TM. Indeed, the conditions (2.2) and (2.3) being fulfiled for $p^i_j(x,y)$, we obtain

$$\begin{cases} \overset{\alpha}{\tilde{P}}{}^3 - \tilde{P} = o, \quad \text{rank } \|\overset{\alpha}{\tilde{P}}(x,y)\|_{\mathcal{H}TM} = \text{rank } \|\overset{\alpha}{\tilde{P}}(x,y)\|_{VTM} = n - k \\ H = \overset{\alpha}{\tilde{P}}{}^2 = h^i_j \, \delta_i \otimes dx^j + h^i_j \, \dot{\partial}_i \otimes \delta y^j , \\ V = - \overset{\alpha}{\tilde{P}}{}^2 + I = v^i_j \, \delta_i \otimes dx^j + v^i_i \, \dot{\partial}_i \otimes \delta y^j, \quad \forall \alpha = 1,2,3 , \end{cases} \tag{5.7}$$

where $h^i_j = p^i_h \, p^h_j$, $v^i_j = - p^i_h \, p^h_j + \delta^i_j$.

Then (5.6) determines $\tilde{P}(3,-1)$-structures on TM by the lifting of one $p(3,-1)$-Finsler sturctures from M to the space total TM of a tangent budle $T(M)$.

We remark that the Nijenhuis tensor of $\tilde{P} \in \tau^1_1(TM)$ is given by

$$\tilde{N}(X,Y) = [\tilde{P}X,\tilde{P}Y] - \tilde{P}[\tilde{P}X,Y] - \tilde{P}[X,\tilde{P}Y] + H[X,Y] , \tag{5.8}$$

and the integrability condition of a $\tilde{P}(3,-1)$-structures is $\tilde{N}(X,Y) = o$, $\forall\, X,Y \in (TM)$. It is sufficient to calculate $\tilde{N}(\delta_j,\delta_k)$, $N(\delta_j,\dot{\partial}_k)$ and $N(\dot{\partial}_j,\dot{\partial}_k)$ and we can determine $\tilde{N}(X,Y)$.

6. The integrability of the $p(3,-1)$-Finsler structures

Let N be a non-linear connection of $T(M)$. Then the $p(3,-1)$-Finsler structure on the base manifold M is lifted to a $\tilde{P}(3,-1)$-structure on $T(M)$ in three manner (5.6). The values of the Finsler components of $\tilde{P}$ from (5.2) are given in the following table:

$\tilde{P}$	$\overset{1}{P}{}^i_j$	$\overset{2}{P}{}^i_j$	$\overset{3}{P}{}^i_j$	$\overset{4}{P}{}^i_j$
$\overset{1}{\tilde{P}}$	p^i_j	0	0	p^i_j
$\overset{2}{\tilde{P}}$	p^i_j	0	0	$-p^i_j$
$\overset{3}{\tilde{P}}$	0	p^i_j	p^i_j	0

We remark the following relations:

$$\begin{cases} \overset{1}{\tilde{P}}(\delta_j) = p^i_j\,\delta_i, & \overset{2}{\tilde{P}}(\delta_j) = p^i_j\,\delta_i, & \overset{3}{\tilde{P}}(\delta_j) = p^i_j\,\dot{\partial}_i, \\ \overset{1}{\tilde{P}}(\dot{\partial}_j) = p^i_j\,\dot{\partial}_i, & \overset{2}{\tilde{P}}(\dot{\partial}_j) = -p^i_j\,\dot{\partial}_i, & \overset{3}{\tilde{P}}(\dot{\partial}_j) = p^i_j\,\delta_i. \end{cases}$$

Definition 6.1. *A p(3,-1) structure of index k on a differentiable manifold M is called an integrable of type I, II or III with respect to the non-linear connection N, if the corresponding lifted* $\overset{1}{\tilde{P}}(3,-1)$, $\overset{2}{\tilde{P}}(3,-1)$, *or* $\overset{3}{\tilde{P}}(3,-1)$*-structure are integrable.*

We characterize these cases of integrability, using only the invariants of the group G_p.

Theorem 6.1. *The p(3,-1)-Finsler structure* $p^i_j(x,y)$, (2.1) *is an integrable of type I, if and only if the following invariant Finsler tensor fields vanish:*

$$(6.2) \qquad \overset{1}{T}{}^{i}_{jk} = o\ ,\quad \overset{1}{R}{}^{i}_{jk} = o\ ,\quad \overset{1}{C}{}^{i}_{jk} = o\ ,\quad \overset{1}{P}{}^{i}_{jk} = o \quad (\Rightarrow \overset{1}{S}{}^{i}_{jk} = o)\ .$$

Proof. The $p(3,-1)$-Finsler structure is an integrable of type I if and only if $\tilde{N}(X,Y) = 0$ for $\overset{1}{\tilde{P}}$. But $N(X,Y) = 0\ \forall\ X,Y \in \mathcal{X}(TM)$ is equivalent to

$$\tilde{N}(\delta_j,\delta_k) = o\ ,\quad \tilde{N}(\delta_j,\dot{\partial}_k) = o\ ,\quad \tilde{N}(\dot{\partial}_j,\dot{\partial}_k) = o$$

which are equivalent to (6.2). In this case, because $\overset{1}{C}{}^{i}_{jk}=o$ we have $\overset{1}{S}{}^{i}_{jk}=o$.

In the same way we can prove

Theorem 6.2. *The $p(3,-1)$-Finsler structure $p^{i}_{j}(x,y)$, (2.1), is an integrable of type II, if and only if the following ivariant Finsler tensor fields vanish:*

$$(6.3) \qquad \overset{1}{T}{}^{i}_{jk} = 0\ ,\quad \overset{2}{R}{}^{i}_{jk} = 0\ ,\quad \overset{2}{C}{}^{i}_{jk} = 0\ ,\quad \overset{2}{P}{}^{i}_{jk} = 0\ ,\quad \overset{1}{S}{}^{i}_{jk} = 0\ .$$

Theorem 6.3. *The $p(3,-1)$-Finsler structure $p^{i}_{j}(x,y)$, (2.1) is an integrable of type III, if and only if the following invariant Finsler tensor fields vanish:*

$$(6.4) \qquad \overset{3}{T}{}^{i}_{jk} = 0\ ,\quad \overset{3}{R}{}^{i}_{jk} = 0\ ,\quad \overset{3}{C}{}^{i}_{jk} = 0\ ,\quad \overset{3}{P}{}^{i}_{jk} = 0\ ,\quad \tilde{T}^{i}_{jk} = 0\ ,\quad \overset{3}{S}{}^{i}_{jk} = 0\ .$$

Concluding by Theorems 6.1, 6.2 and 6.3 we obtain:

Theorem 6.4. *The $p(3,-1)$-Finsler structure $p^{i}_{j}(x,y)$, (2.1), is an integrable of type I, II or III if and only if the invariants of the group G_p have the values given in the following table:*

Type of integrability	Characterization by invariants
I	$\overset{1}{T}{}^{i}_{jk} = 0,\ \overset{1}{R}{}^{i}_{jk} = 0,\ \overset{1}{C}{}^{i}_{jk} = 0,\ \overset{1}{P}{}^{i}_{jk} = 0,\ \overset{1}{S}{}^{i}_{jk} = 0\ .$
II	$\overset{1}{T}{}^{i}_{jk} = 0,\ \overset{2}{R}{}^{i}_{jk} = 0,\ \overset{2}{C}{}^{i}_{jk} = 0,\ \overset{2}{P}{}^{i}_{jk} = 0,\ \overset{1}{S}{}^{i}_{jk} = 0\ .$
III	$\overset{3}{T}{}^{i}_{jk} = 0,\ \overset{3}{R}{}^{i}_{jk} = 0,\ \overset{3}{C}{}^{i}_{jk} = 0,\ \overset{3}{P}{}^{i}_{jk} = 0,\ \overset{3}{S}{}^{i}_{jk} = 0,\ \tilde{T}^{i}_{jk}=0$

7. (h,v)-Walker structure on the total space of a vector bundle

Let $\xi = (E,\pi,M)$ a vector bundkle of the class C^{∞}, let N be a non-linear connection on E and let us denote by HE and VE the complementary horizontal and vertical distribution:

$$T_{(x,y)} E = H_{(x,y)} E + V_{(x,y)} E , \quad \forall\ (x,y) \in E .$$

Let $l(x,y)$ be an $l(3,-1)$-structure of index k_1 on the distribution HE and let $m(x,y)$ be an $m(3,-1)$-structure of index k_2 on the distribution VE. For any $(x,y) \in E$ we have

(7.1) $$HE = H_h E + H_v E \qquad VE = V_h E + V_v E .$$

We denote by $\underset{1}{h}$, $\underset{1}{v}$ and respectively $\underset{2}{h}$, $\underset{2}{v}$, the supplementary projectors on the distribution (7.1).

Let $\{\delta_i , \dot{\partial}_a\}$, $\{dx^i , \delta y^a\}$ be the adapted basis of N and VE.

Under these conditions we can consider on E the aggregate tensor field of type $(1,1)$ given by

(7.2) $$P = l^i_j(x,y)\delta_i \otimes dx^j + m^a_b(x,y)\, \dot{\partial}_a \otimes \delta y^b .$$

Because (2.2) and (2.3), § 2, are fulfilled for l, $\underset{1}{h}$, $\underset{1}{v}$ and respectively m, $\underset{2}{h}$, $\underset{2}{v}$ we have from (7.2) the following relations:

(7.3) $$P^3 - P = 0 , \quad \text{rank}\, \|P\|_{HE} = k_1 , \quad \text{rank}\, \|P\|_{VE} = k_2 .$$

We put $H = P^2$, $V = -P^2 + I$ and we obtain

(7.4) $$\begin{cases} H = \underset{1}{h}{}^i_j(x,y)\ \delta_i \otimes dx^j + \underset{2}{h}{}^a_b(x,y)\ \dot{\partial}_a \otimes \delta y^b , \\ V = \underset{1}{v}{}^i_j(x,y)\ \delta_i \otimes dx^j + \underset{2}{v}{}^a_b(x,y)\ \dot{\partial}_a \otimes \delta y^b , \end{cases}$$

An elementary calculus shows us that

(7.5) $$\begin{cases} HP = PH = P , \quad VP = PV = 0 , \\ P^2H = H \quad , \quad P^2V = 0 . \end{cases}$$

This means that (7.2) is a $P(3,-1)$-structure on E.

Definition 7.1. *We shall call the structure given by (7.2) the (h,v)-Walker structure on the total space E of a vector bundle ξ.*

Definiton 7.2. *A d-connection D on E with coefficients* $D\Gamma = \left(L^i_{jk},\ L^a_{bk},\ C^i_{jc},\ C^a_{bc}\right)$ *is called a Walker d-connection or compatible with the (h,v)-Walker structure* $P \in \tau^1_1(E)$, (6.2), *if*

(7.6) $$l^i_{j|k} = 0, \quad m^a_{b|k}, \quad l^i_j\big|_c = 0, \quad m^a_b\big|_c = 0,$$

where $|$ *and* $\big|$ *denotes h-and v-covariant derivatives with respect to* $D\Gamma$.

Theorem 7.1. *Let* $D\overset{\circ}{\Gamma} = \left(\overset{\circ}{L}{}^i_{jk},\ \overset{\circ}{L}{}^a_{bk},\ \overset{\circ}{C}{}^i_{jc},\ \overset{\circ}{C}{}^a_{bc}\right)$ *be a fixed d-connection. Then, the d-connection* $D\overset{*}{\Gamma}$ *given by*

(7.7) $$\begin{cases} \overset{*}{L}{}^i_{jk} = \overset{\circ}{L}{}^i_{jk} + \dfrac{1}{2}\left\{l^i_s\, l^s_{j\overset{\circ}{|}k} - \underset{1}{v}{}^i_{j\overset{\circ}{|}k} + 3\underset{1}{v}{}^i_s\, \underset{1}{v}{}^s_{j\overset{\circ}{|}k}\right\}, \\ \overset{*}{L}{}^a_{bk} = \overset{\circ}{L}{}^a_{bk} + \dfrac{1}{2}\left\{m^a_f\, m^f_{b\overset{\circ}{|}k} - \underset{2}{v}{}^a_{b\overset{\circ}{|}k} + 3\underset{2}{v}{}^a_f\, \underset{2}{v}{}^f_{b\overset{\circ}{|}k}\right\}, \\ \overset{*}{C}{}^i_{jc} = \overset{\circ}{L}{}^i_{jc} + \dfrac{1}{2}\left\{l^i_s\, l^s_j\overset{\circ}{\Big|}_c - \underset{1}{v}{}^i_j\overset{\circ}{\Big|}_c + 3\underset{1}{v}{}^i_s\, \underset{1}{v}{}^s_j\overset{\circ}{\Big|}_c\right\}, \\ \overset{*}{C}{}^a_{bc} = \overset{\circ}{C}{}^a_{bc} + \dfrac{1}{2}\left\{m^a_f\, m^f_b\overset{\circ}{\Big|}_c - \underset{2}{v}{}^a_b\overset{\circ}{\Big|}_c + 3\underset{2}{v}{}^a_f\, \underset{2}{v}{}^f_b\overset{\circ}{\Big|}_c\right\}, \end{cases}$$

is a Walker d-connection, where $\overset{\circ}{|}$ *and* $\overset{\circ}{\Big|}$ *are the h-and v-covariant derivatives with respect to* $D\overset{\circ}{\Gamma}$.

If we take in (6.7) for $D\overset{\circ}{\Gamma}$ a Berward connection

(7.8) $$\overset{\circ}{L}{}^a_{bk} = \dot{\partial}_b N^a_k, \quad \overset{\circ}{C}{}^i_{jc} = 0,$$

the we have

Theorem 7.2. *Let* $D\overset{\circ}{\Gamma}$ *be a fixed Berward connection. Then, the d-connection* $D\overset{\gamma}{\Gamma}$ *given by*

(7.9) $$\begin{cases} \overset{\gamma}{L}{}^i_{jk} = \overset{\circ}{L}{}^i_{jk} + \dfrac{1}{2}\left\{l^i_s\, l^s_{j\overset{\circ}{|}k} - \underset{1}{v}{}^i_{j\overset{\circ}{|}k} + 3\underset{1}{v}{}^i_s\, v^s_{j\overset{\circ}{|}k}\right\} \quad (=\overset{*}{L}{}^i_{jk}), \\ \overset{\gamma}{L}{}^a_{bk} = \dot{\partial}_b N^a_k + \dfrac{1}{2}\left\{m^a_f\left[\delta_k m^f_b + m^d_b\, \dot{\partial}_d\, N^f_k - m^f_d\, \dot{\partial}_b\, N^d_k\right] - \right. \\ \qquad \left. - \left[\delta^a_f - 3\,\underset{2}{v}{}^a_f\right]\left[\delta_k\, \underset{2}{v}{}^f_b + \underset{2}{v}{}^d_b\, \dot{\partial}_d\, N^f_k - \underset{2}{v}{}^f_d\, \dot{\partial}_b\, N^d_k\right]\right\}, \\ \overset{\gamma}{C}{}^i_{jc} = \dfrac{1}{2}\left\{l^i_s\, \dot{\partial}_c\, l^s_j - \left[\delta^i_s - 3\underset{1}{v}{}^i_s\right]\dot{\partial}_c\, \underset{1}{v}{}^s_j\right\}, \\ \overset{\gamma}{C}{}^a_{bc} = \overset{\circ}{C}{}^a_{bc} + \dfrac{1}{2}\left\{m^a_f\, m^f_b\overset{\circ}{\Big|}_c - \underset{2}{v}{}^a_b\overset{\circ}{\Big|}_c + 3\underset{2}{v}{}^a_f\, \underset{2}{v}{}^f_b\overset{\circ}{\Big|}_c\right\} \quad (= \overset{*}{C}{}^a_{bc}), \end{cases}$$

is a Walker d-connection, with respect to $p \in \tau^1_1(E)$, (7.2).

Remark 7.1. In (7.9) the coefficients $\overset{\vee}{L}{}^a_{bk}$, and $\overset{\vee}{C}{}^i_{jc}$ of the Walker d-connection $D\overset{\vee}{\Gamma}$ depend on $N^i_j(x,y)$ $l^i_j(x,y)$ and $m^a_b(x,y)$, only, that is, they are canonically introduced.

If we denote Obata's operators (2.5), 2, of $l^i_j(x,y)$ and $m^a_b(x,y)$ by $\underset{1}{A}{}^{rs}_{ij}$, $\underset{2}{A}{}^{rs}_{ij}$ and respectively $\underset{1}{B}{}^{cd}_{ab}$, $\underset{2}{B}{}^{cd}_{ab}$, it is easy to prove:

Theorem 7.3. *The set of all the Walker d-connections with respect to the Walker structure* $P \in \tau^1_1(E)$, *(7.2),* is given by

$$(7.10)\begin{cases} L^i_{jk} = \overset{*}{L}{}^i_{jk} + \underset{1}{A}{}^{is}_{rj} X^r_{sk}, & L^a_{bk} = \overset{*}{L}{}^a_{bk} + B^{af}_{db} X^d_{fk}, \quad \forall\, X^i_{jk} \in \tau^{10}_{20}, \quad X^a_{bk} \in \tau^{01}_{11}, \\ C^i_{jc} = \overset{*}{C}{}^i_{jc} + \underset{1}{A}{}^{is}_{rj} Y^r_{sk}, & C^a_{bc} = \overset{*}{C}{}^a_{bc} + B^{af}_{db} Y^d_{fc}, \quad \forall\, Y^i_{jc} \in \tau^{10}_{11}, \quad Y^a_{bc} \in \tau^{01}_{02}, \end{cases}$$

The transformations of Walker d-connections $D\Gamma(N) \to D\Gamma(\bar{N})$ with the same non-linear connection N are given by

$$(7.11)\qquad \begin{cases} \bar{L}{}^i_{jk} = L^i_{jk} + \underset{1}{A}{}^{is}_{rj} X^r_{sk}, & \bar{L}{}^a_{bk} = L^a_{bk} + \underset{1}{B}{}^{af}_{db} X^d_{fk}, \\ \bar{C}{}^i_{jc} = C^i_{jc} + \underset{1}{A}{}^{is}_{rj} Y^r_{sc}, & \bar{L}{}^a_{bc} = L^a_{bc} + \underset{1}{B}{}^{af}_{db} X^d_{fc}, \end{cases}$$

Hence,

Theorem 7.4. *The set of all transformations (7.11) and the mapping product is an Abelian group* G_W *isommorphic with the additive group of a d-tensor field of the form*

$$\left(\underset{1}{A}{}^{is}_{rj} X^r_{sk},\ \underset{1}{B}{}^{af}_{db} X^d_{fk},\ \underset{1}{A}{}^{is}_{rj} Y^r_{sc},\ \underset{1}{B}{}^{af}_{db} Y^d_{fc} \right).$$

Theorem 7.5. *The following d-tensor fields are invariants by the action of the group* G_W :

$$(7.12)\begin{cases} T(P)^i_{jk} = \underset{1}{h}{}^i_m T^m_{jk} + l^r_j\, l^s_k\, T^i_{rs} - \left(l^r_j\, T^m_{rk} + l^s_k\, T^m_{js} \right) l^i_m \quad (= \overset{1}{T}(l)^i_{jk}), \\ R(P)^a_{jk} = \underset{2}{h}{}^a_d R^d_{jk} + l^r_j\, l^s_k\, R^a_{rs} - \left(l^r_j\, R^d_{rk} + l^s_k\, R^d_{js} \right) m^a_d, \\ C(P)^i_{jk} = \underset{1}{h}{}^i_m C^m_{jc} + l^r_j\, m^g_c\, C^i_{rg} - \left(l^r_j\, C^m_{rc} + m^g_c\, C^m_{jg} \right) l^i_m, \\ P(P)^a_{jc} = \underset{2}{h}{}^a_d P^d_{jc} + l^r_j\, m^g_c\, P^a_{rg} - \left(l^r_j\, P^d_{rc} + m^g_c\, P^d_{jg} \right) m^a_d, \\ S(P)^a_{bc} = \underset{2}{h}{}^a_d S^d_{bc} + m^f_b\, m^g_c\, S^a_{fg} - \left(m^f_b\, S^d_{fc} + m^g_c\, S^d_{bd} \right) m^a_d \quad (= \overset{1}{S}(m)^a_{bc}). \end{cases}$$

Theorem 7.6. *The d-tensor fields* $T(P)^i_{jk}$, $S(P)^a_{bc}$ *vanish if and only if there exists on E an h-and v-semi-symmetric Walker d-connection* $D\Gamma(N)$.

The integrability condition of the (h,v)-Walker stucture $P \in \tau^1_1(E)$, (6.2), is $N(P)(X,Y) = o$, $\forall\ X,Y \in \mathcal{X}(E)$. Taking into account that we have the following relationships:

$$P(\delta_j) = l^i_j\ \delta_i\ , \quad P(\dot{\partial}_b) = m^a_b\ \dot{\partial}_a\ , \tag{7.13}$$

we obtain

Theorem 7.7. *The (h,v)-Walker structure* $P \in \tau^1_1(E)$, (6.2), *is integrable if and only if the invariants* (7.12) vanish:

$$T(P)^i_{jk} = o,\quad R(P)^a_{jk} = o,\quad C(P)^i_{jc} = o,\quad P(P)^a_{jc} = o,\quad S(P)^a_{bc} = o\ . \tag{7.14}$$

Remark 7.2. If $E = TM$, the considerations from this paragraph remain valid.

References

1. **Atanasiu, Gh.:** f(3,1)-structures: the Finsler case and the vector bundle case (to appear).
2. **Atanasiu, Gh., and Hashiguchi, M.:** Almost tangent Finsler structures and almost tangent Finsler connections, to appear in Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.).
3. **Atanasiu, Gh., and Klepp, C.:** Almost product Finsler structures and connections, Studia Sci. Math. Hungarica, Budapest, 18 (1983), 43-56.
4. **Atanasiu, Gh., and Klepp, C.:** p-structures of constant rank, Lucr. Sem. Matem. - Fizica, I. Politehnic "Traian Vuia" Timişoara, (1987) (to appear).
5. **Atanasiu, Gh., and Klepp, C.:** P-Finsler connections, Proc. Symp. Math. and Appl., vol. II, Acad. R.S. Romania, Sect. Timişoara, (1987) (to appear).
6. **Atanasiu, Gh., Hasniguchi, M. and Miron, R.:** Supergeneralized Finsler spaces, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) 18 (1985), 19-34.
7. **Klepp, C.:** Finsler geometry on vector bundles, Anal. St. Univ. "Al. I. Cuza" Iaşi, Supl. t. XXVII, s. I a (1981), 37-42.
8. **Klepp, C.:** Almost product Finsler structures, Anal. St. Univ "Al. I. Cuza" Iaşi, Supl. t XXVIII, s.I. a (1982), 59-67.

9. **Klepp, C.:** Metrical almost product Finsler structures, Anal. St. Univ. "Al. I. Cuza" Iaşi, t XXIX, s. I a Matematica, f.2 (1983), 21-25.

10. **Klepp, C.:** Almost product Finsler structures and connections on vector bundle, Proc. of the third Nat. Sem. Finsler Spaces, University of Brasov (1984), 105-115.

11. **Matsumoto, M.:** Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, Otsu, Japan, 1986.

12. **Miron, R.:** Introduction to the theory of Finsler spaces, Proc. Nat. Sem. Finsler Spaces, University of Brasov, vol.I (1980), 131-183.

13. **Miron, R.:** On trasformation groups of Finsler connections, Tensor, N.s,35 (1981), 235-240.

14. **Miron, R.:** On almost complex Finsler structures, Anal. Univ. "Al.I. Cuza" Iaşi, t XXVIII, s. I a, f.2 (1982), 13-17.

15. **Miron, R.:** Vector bundles. Finsler geometry, Proc. Nat. Sem. Finsler Spaces, University of Braşov, vol. II (1982), 147-188.

16. **Miron, R.:** Techniques of Finsler geometry in the theory of vector bundles, Acta Sci. Math., Szeged, 49, 1-4 (1985), 119-129.

17. **Miron, R. şi Anastasiei, M:** Fibrate vectoriale. Spatii Lagrange. Aplicatii in teoria relativitatii. Ed. Acad. R. S. România, Bucureşti, 1987.

18. **Miron, R et Atanasiu, Gh.:** Sur les (f,g)-connecions linéaires et l'integrabilité des (f,g)-structures, Bull. Math. de la Soc. Sci. Math. R.S.R., t.30(78), Nr.3(1986), 245-256.

19. **Obata, M.:** Affine connections on manifolds with almost complex, quaternion or Hermitian structures, Jap. J. Math., 26(1957), 43-77.

20. **Oproiu, V.:** Some remarkable structures and connections defined on the tangent bundles, Rendiconti di Matem., (3) vol. 6, serie VI (1973), 503-540.

21. **Oproiu, V.:** Some properties of the tangent bundles related to the Finsler geometry, Proc. Nat. Sem. Finsler Spaces, University of Braşov, vol.I (1980), 195-207.

22. **Petrakis, A.:** Rapport entre le tenseur N de Nijenhuis et le tenseur T de la torsion d'une f-structure qui satisfait la relation $f^3+f = 0$, Tensor N.S., 36, 3 (1982), 312-314.

23. **Stoica, E.:** A geometrical characterization of normal Finsler connection Anal.St.Univ. "Al.I. Cuza", Iaşi, vol.XXX, s.I a (1984), 3.

24. **Udriste, C.:** Diagonal lifts from a manifold to its tangent bundle, Rendiconti di Matem. (4), vol.9, serie VI (1976), 539-550.

25. **Vamanu, E.:** Connections and almost-product structure, Bul. Inst. Politehn. Iaşi Sect. I, 24 (28) (1978), 39-45.

26. **Vamanu, E., et Atanasiu, Gh.:** Sur les transformations de connexions presque produit, Bul. Univ. Braşov, Ser. C. tXXII (1980), 173-180.

27. **Walker, A.G.,** Connections for parallel distributions in the large I; II, Quart J. Math. Oxford, Ser. (2) (1955), 301-308; Ser.(2) 9 (1958), 221-231.

28. **Walker, A.G.:** Almost-product structure, Proc. of the Third. Symp. in Pure Math. American Math. Soc., 3 (1961), 94-100.

Rezime

$p(3,-1)$ - FINSLEROVE STRUKTURE I NJIHOVI LIFTOVI

Na diferencijalnoj mnogostrukosti M kao i na TM je definisana $p(3,-1)$ odnosno $P(3,-1)$ Finslerova struktura. Odredeni su koeficijenti koneksije koje su saglasne sa tim strukturama, dati su uslovi integrabilnosti. Nadene su invarijante grupe transformacije Finslerove koneksije i dati su neki specijalni slučajevi.

Received by the editors, March 12, 1988.