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SOME THEOREMS ON CONFORMALLY QUASI-RECURRENT MANIFOLDS

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Abstract

A conformally quasi-recurrent manifold is an n-dimensional ($n > 3$) Riemannian manifold whose conformal curvature tensor satisfies the condition (1.2), where ∇ is the operator of covariant differentiation. It is proved that if a_i is a gradient vector field, such a manifold can be conformally related to the conformally symmetric one (i.e. to the manifold satisfying $\nabla_s C_{hijk} = 0$). Using this fact, it is proved that many properties of conformally symmetric manifolds can be generalized in such a manner that they hold good for conformally quasi-recurrent manifolds too (in which a_i is a gradient vector field). Also, some properties of general quasi-recurrent manifolds are obtained.

1. Introduction

A conformally quasi-reccurent manifold has been defined in [3] as an n-dimensional ($n > 3$) Riemannian manifold M with a (possibly indefinite) metric g whose conformal curvature tensor

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(g_{ij}R^h_k - g_{ik}R^h_j + \delta^h_k R_{ij} - \delta^h_j R_{ik})$$

$$+ \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{1.1}$$

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satisfies the condition

(1.2) $$\nabla_s C_{hijk} = 2a_s C_{hijk} + a_h C_{sijk} + a_i C_{hsjk} + a_j C_{hisk} + a_k C_{hijs}$$

where

$$C_{hijk} = g_{ht} C^t_{ijk} ,$$

a_s is a vector field and R^h_{ijk}, R_{ij}, R and ∇ denote the curvature tensor, Ricci tensor, scalar curvature and covariant differentiation respectively.

If

(1.3) $$\nabla_s C_{hijk} = 0,$$

M is said to be conformally symmetric. If, however,

$$\nabla_s C_{hijk} = a_s C_{hijk} ,$$

M is said to be conformally recurrent. Therefore, a conformally symmetric manifold is a special case of both the conformally recurrent and the conformally quasi-recurrent one.

M is said to be essentially conformally symmetric if it satisfies (1.3) but is neither conformally flat nor locally symmetric. These manifolds have been investigated in detail in [1], [2], [4] and [5]. Among other things it is proved there that any essentially conformally symmetric manifold satisfies the following relations:

(a) $\nabla_j R_{ik} = \nabla_k R_{ij}$;

(b) $R_{aj} R^a_{ikl} + R_{ak} R^a_{ilj} + R_{al} R^a_{ijk} = 0$;

(c) $R_{aj} C^a_{ikl} + R_{ak} C^a_{ilj} + R_{al} C^a_{ijk} = 0$;

(d) $\nabla_s R_{aj} C^a_{ikl} + \nabla_s R_{ak} C^a_{ilj} + \nabla_s R_{al} C^a_{ijk} = 0$;

(e) $R = 0$;

(f) $R_{la} C^a_{jkl} = 0$;

(g) $C_{lmh}{}^{a} C_{aijk} + C_{lmi}{}^{a} C_{hajk} + C_{lmj}{}^{a} C_{hiak} + C_{lmk}{}^{a} C_{hija} = 0$;

(h) $\{R_{hl} C_{mijk} + R_{il} C_{hmjk} + R_{jl} C_{himk} + R_{kl} C_{hijm}\} - \{m/l\} = 0$, where in the expression $\{\dots\} - \{m/l\}$, $\{m/l\}$ means the first bracket in which m and l interchange their places;

(i) $R^a_i R_{aj} = 0$;

(j) $R_{ij} R_{hk} - R_{hj} R_{ik} = F\, C_{hijk}$, for some function F.

The purpose of this paper is to find relations correspondig to (a)-(j) for a conformally quasi-recurrent manifold. In §2 we shall obtain the relations corresponding to (a) and (d) and prove that (b) and (c) hold good. In §3 and §4 we deal with conformally quasi-recurrent manifolds in which the vector field a_i is a gradient. In §3 we shall prove that such a manifold can be conformally related to the conformally symmetric one. Using this, we shall find in §4 the relations corresponding to (e)-(j).

2. General case

It is well known that the conformal curvature tensor satisfies the relations

(2.1) $$C^h_{ijk} = - C^h_{ikj} ,$$

(2.2) $$C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0 ,$$

(2.3) $$C^a_{ajk} = C^a_{jak} = C^a_{jka} = 0 ,$$

(2.4) $$C_{hijk} = - C_{ihjk} , \quad C_{hijk} = C_{jkhi} .$$

Now, we shall prove

Theorem 1. *Any conformally quasi-recurrent manifold satisfies*

(2.5) $$\nabla_k \Pi_{ij} = \nabla_j \Pi_{ik} = 0 ,$$

where

(2.6) $$\Pi_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij} .$$

Relation (2.5) corresponds to relation (a) of §1.

Proof. Transvecting (1.2) with g^{hk} and taking into account (2.3), we get

$$a^h C_{sijh} + a^h C_{hijs} = 0$$

which, because of (2.1) and (2.4), can be rewritten in the form :

$$a^h C_{hijs} = a^h C_{hjsi} \; . \tag{2.7}$$

On the other hand, in view of (2.2), we have

$$a^h (C_{hijs} + C_{hjsi} + C_{hsij}) = 0 \; ,$$

which, using (2.7), reduces to

$$a^h C_{hijs} = 0 \; . \tag{2.8}$$

Therefore, transvecting (1.2) with g^{sh}, we get

$$\nabla_s C^s_{\ ijk} = 0 \; . \tag{2.9}$$

On the other hand, differentiating (1.1) covariantly, we have

$$\nabla_s C^h_{\ ijk} = \nabla_s R^h_{\ ijk} - \frac{1}{(n-2)} (g_{ij} \nabla_s R^h_k - g_{ik} \nabla_s R^h_j + \delta^h_k \nabla_s R_{ij} - \delta^h_j \nabla_s R_{ik}) + $$
$$+ \frac{\nabla_s R}{(n-1)(n-2)} (\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

from which, contracting with respect to s and h and taking into account that

$$\nabla_s R^s_{\ ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} \quad \text{and} \quad \nabla_s R^s_k = \frac{1}{2} \nabla_k R \; ,$$

we get

$$\nabla_s C^s_{\ ijk} = \frac{n-3}{n-2} \left[\nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (g_{ij} \nabla_k R - g_{ik} \nabla_j R) \right] \; .$$

This, together with (2.9) and (2.6) leads to (2.5), because of $n > 3$.

Theorem 2. *For any conformally quasi-recurrent manifold, relations (b) and (c) of §1 hold good.*

Proof. Differentiating (2.5) covariantly, we get

$$\nabla_l \nabla_k \Pi_{ij} - \nabla_l \nabla_j \Pi_{ik} = 0 \; . \tag{2.10}$$

Permuting in (2.10) the indices j, k and l cyclically, adding the resulting equations to (2.10) and using the Ricci identity, we obtain

$$\Pi_{aj}R^{a}{}_{ikl} + \Pi_{ak}R^{a}{}_{ilj} + \Pi_{al}R^{a}{}_{ijk} = 0 . \tag{2.11}$$

Substituting (2.6) into this relation, we get (b) of §1.

On the other hand, (1.1) can be written in the form

$$R^{h}{}_{ijk} = C^{h}{}_{ijk} + \frac{1}{n-2}\Big[g_{ij}\Big(R^{h}_{k} - \frac{R}{2(n-1)}\delta^{h}_{k}\Big) - g_{ik}\Big(R^{h}_{j} - \frac{R}{2(n-1)}\delta^{h}_{j}\Big) +$$

$$+ \delta^{h}_{k}\Big(R_{ij} - \frac{R}{2(n-1)}g_{ij}\Big) - \delta^{h}_{j}\Big(R_{ik} - \frac{R}{2(n-1)}g_{ik}\Big)\Big]$$

or, using (2.6), in the form

$$R^{h}{}_{ijk} = C^{h}{}_{ijk} + \frac{1}{n-2}\Big(g_{ij}\Pi^{h}_{k} - g_{ik}\Pi^{h}_{j} + \delta^{h}_{k}\Pi_{ij} - \delta^{h}_{j}\Pi_{ik}\Big)$$

Substituting this into (2.11) and taking into account that Π_{ij} is a symmetric tensor, we get

$$\Pi_{aj}C^{a}{}_{ikl} + \Pi_{ak}C^{a}{}_{ilj} + \Pi_{al}C^{a}{}_{ijk} = 0 .$$

Substituting (2.6) into this relation, we obtain (c) of §1.

Theorem 3. *Any conformally quasi-recurrent manifold satisfies*

$$T_{saj}C^{a}{}_{ikl} + T_{sak}C^{a}{}_{ilj} + T_{sal}C^{a}{}_{ijk} = 0 , \tag{2.12}$$

where

$$T_{saj} = \nabla_{s}R_{aj} - R_{sa}a_{j} + g_{sa}a^{t}R_{tj} .$$

Relation (2.12) corresponds to (f) of §1.

Proof. Differentiating relation (c) of §1 covariantly, substituting (1.2) and using (c), we find

$$(\nabla_{s}R_{aj})C^{a}{}_{ikl} + (\nabla_{s}R_{ak})C^{a}{}_{ilj} + (\nabla_{s}R_{al})C^{a}{}_{ijk} +$$

$$+ a^{t}(R_{tj}C_{sikl} + R_{tk}C_{silj} + R_{tl}C_{sijk}) +$$

$$+ a_k(R_{aj}C^a_{isl} + R_{al}C^a_{ijs}) + a_l(R_{aj}C^a_{iks} + R_{ak}C^a_{isj}) +$$

$$+ a_j(R_{ak}C^a_{ils} + R_{al}C^a_{isk}) = 0 .$$

Using (c) once more, we obtain (2.12).

In the sequel, we shall need

Lemma *Any conformally quasi-recurrent manifold satisfies*

$$(\nabla_r a_h)C^h_{ijk} = - a_h a^h C_{rijk} . \tag{2.13}$$

Proof. Differentiating (2.8) covariantly, we have

$$(\nabla_r a_h)C^h_{ijk} + a_h \nabla_r C^h_{ijk} = 0 .$$

Now, substituting (1.2) and using (2.8), we get (2.13).

3. Conformal change of a conformally quasi-recurrent manifold in which a_i is a gradient vector field

Now, let us suppose that for each point $x \in M$ of conformally quasi-recurrent manifold (M, g) there exists a neighbourhood U of x and a function f on U such that $a_i = \dfrac{\partial f}{\partial x^i}$. Let metrics $\bar{g}$ and g be conformally related such that

$$\bar{g}_{ij} = e^{2f} g_{ij} , \quad \bar{g}^{ij} = e^{-2f} g^{ij} . \tag{3.1}$$

Then, the Christoffel symbols of metrics $\bar{g}$ and g are related as follows:

$$\overline{\left\{ {k \atop ij} \right\}} = \left\{ {k \atop ij} \right\} + \delta^k_i a_j + \delta^k_j a_i - g_{ij} a^k ,$$

while the conformal curvature tensor is invariant:

$$\bar{C}^h_{ijk} = C^h_{ijk} . \tag{3.2}$$

As for tensor $\bar{C}_{hijk}$, we have

$$\bar{C}_{hijk} = e^{2f} C_{hijk} . \tag{3.3}$$

Let $\bar{\nabla}$ be the operator of covariant differentiation with respect to $\overline{\left\{ {k \atop ij} \right\}}$. Then, applying it to (3.2), and using (3.1), we get

$$\bar{\nabla}_s \bar{C}^h_{ijk} = \nabla_s C^h_{ijk}$$

(3.4) $$- 2 a_s C^h_{ijk} - a^h C_{sijk} - a_i C^h_{sjk} - a_j C^h_{isk} - a_k C^h_{ijs}$$

$$+ \delta^h_s a_r C^r_{ijk} + g_{is} a^r C^h_{rjk} + g_{js} a^r C^h_{irk} + g_{ks} a^r C^h_{ijr} ,$$

from which follows

(3.5) $$\bar{\nabla}_s \bar{C}^h_{ijk} = 0 ,$$

because of (1.2), (2.1), (2.4) and (2.8).

Thus, if (M, g) is conformally quasi-recurrent, then from $a_i = \dfrac{\partial f}{\partial x^i}$ and (3.1) it follows that $(M, \bar{g})$ is conformally symmetric.

Consider, now, instead of (3.1), a new conformal change

$$\tilde{g}_{ij} = e^{2\varphi} g_{ij} , \quad \varphi = \varphi(x^1, \dots, x^n), \quad \frac{\partial \varphi}{\partial x^i} = b_i .$$

As in (3.4), we have

$$\tilde{\nabla}_s \tilde{C}^h_{ijk} = \nabla_s C^h_{ijk}$$

$$- 2 b_s C^h_{ijk} - b^h C_{sijk} - b_i C^h_{sjk} - b_j C^h_{isk} - b_k C^h_{ijs}$$

$$+ \delta^h_s b_r C^r_{ijk} + g_{is} b^r C^h_{rjk} + g_{js} b^r C^h_{irk} + g_{ks} b^r C^h_{ijr} .$$

Suppose that $(M, \tilde{g})$ is conformally symmetric. Then, substituting $\tilde{\nabla}_s \tilde{C}^h_{ijk} = 0$ and (1.2) into the preceding relation, we get

$$2(a_s - b_s) C^h_{ijk} + (a^h - b^h) C_{sijk} + (a_i - b_i) C^h_{sjk} + (a_j - b_j) C^h_{isk}$$

(3.6) $$+ (a_k - b_k) C^h_{ijs}$$

$$+ \delta^h_s b_r C^r_{ijk} + g_{is} b^r C^h_{rjk} + g_{js} b^r C^r_{irk} + g_{ks} b^r C^h_{ijr} = 0 .$$

Contracting (3.6) with respect to h and s and using (2.2), (2.3), (2.4) and (2.8), we obtain

$$(n - 3) b_s C^s_{ijk} = 0 \quad \text{i.e.} \quad b_s C^s_{ijk} = 0$$

because of $n > 3$. Thus, (3.6) can be written in the form

$$(3.7) \qquad 2\nu_s C_{hijk} + \nu_h C_{sijk} + \nu_i C_{hsjk} + \nu_j C_{hisk} + \nu_k C_{hijs} = 0 ,$$

where we have put

$$\nu_i = a_i - b_i .$$

But, it follows from (3.7) that either $\nu_i=0$ or $C_{hijk}=0$ (see [6], Lemma 3). Under our assumption, $C_{hijk}\neq 0$. Therefore, $\nu_i=0$ i.e. $a_i=b_i$. Thus we have proved

Theorem 4. *A conformally quasi-recurrent manifold in which a_i is a gradient vector field, can always be locally conformally related to a conformally symmetric manifold.*

Conversely, if a conformally quasi-recurrent manifold can be conformally related to a conformally symmetric one, and if the corresponding conformal change is of the form (3.1), then

a) the vector field a_i is locally a gradient and $a_i = \dfrac{\partial f}{\partial x^i}$;

b) function f satisfies the condition $a_r \overline{C}^r_{ijk} = 0$.

Statement b) is an immediate consequence of (3.2) and (2.8).

4. Some properties of conformally quasi-recurrent manifolds in which a_i is a gradient vector field

Now, using Theorem 4 and the properties (e)-(j) of a conformally symmetric manifold, it is easy to find the corresponding properties of conformally quasi-recurrent manifolds in which a_i is a gradient vector field.

In all the theorems in this section, a conformally quasi-recurrent manifold means that one in which a_i is a gradient vector field.

As an immediate consequence of Theorem 4, (3.2), (3.3) and (g) of §1, we have

Theorem 5. *Let M be a conformally quasi-recurrent manifold. Then, relation* (*g*) holds good.

To obtain the other properties, we designate by $\overline{R}^h_{ijk}$ the cuvature tensor of a Riemannian space with metric (3.1), i.e. of that conformally symmetric manifold which is conformally related to the considered conformally quasi-recurrent one. Then,

$$\bar{R}^h_{ijk} = R^h_{ijk} \quad \delta^h_k\alpha_{ji} - \delta^h_j\alpha_{ki} + g_{ij}\alpha^h_k - g_{ik}\alpha^h_j , \tag{4.1}$$

where

$$\alpha_{ji} = \nabla_j a_i - a_i a_j + \frac{1}{2} g_{ij} a_t a^t . \tag{4.2}$$

Contracting (4.1) with respect to k and h, we find

$$\bar{R}_{ij} = R_{ij} + (n-2)\alpha_{ji} + g_{ij}\alpha^t_t . \tag{4.3}$$

Transvecting (4.3) with $\bar{g}^{ij}$, we get

$$\bar{R} = e^{-2f} [R + 2(n-1)\alpha^t_t]$$

from which, as a consequence of Theorem 4 and relation (e) of §1, we have

Theorem 6. *The scalar curvature of a conformally quasi-recurrent manifold has the form*

$$R = -2(n-1)\alpha^t_t . \tag{4.4}$$

Using (4.4), we can rewrite (4.3) in the form

$$\bar{R}_{ij} = \Pi_{ij} + (n-2)\alpha_{ij} . \tag{4.5}$$

Also, we can prove

Theorem 7. *Let M be a conformally quasi-recurrent manifold. Then M satisfies the relations.*

$$\Pi_{it}C^t_{jkl} = \frac{n-2}{2} a_t a^t C_{ijkl} ; \tag{4.6}$$

$$R_{it}C^t_{jkl} = -\nabla_t a^t C_{ijkl} ; \tag{4.7}$$

$$R^{ab}C_{ajkb} = 0 ; \tag{4.8}$$

$$\Bigg\{\Big[\Pi_{hl} + (n-2)\alpha_{hl}\Big] C_{mijk} + \Big[\Pi_{il} + (n-2)\alpha_{il}\Big] C_{hmjk} \tag{4.9}$$

$$+\left[\Pi_{jl} + (n-2)\alpha_{jl}\right] C_{hlmk} + \left[\Pi_{kl} + (n-2)\alpha_{kl}\right] C_{hljm}\Big\} - \{m/l\} = 0;$$

(4.10) $$\left[\Pi_{la} + (n-2)\alpha_{la}\right]\left[\Pi^{a}_{j} + (n-2)\alpha^{a}_{j}\right] = 0\ ;$$

(4.11) $$\left[\Pi_{ij} + (n-2)\alpha_{ij}\right]\left[\Pi_{hk} + (n-2)\alpha_{hk}\right] - \left[\Pi_{hj} + (n-2)\alpha_{hj}\right]\left[\Pi_{ik} + (n-2)\alpha_{ik}\right]$$

$$= FC_{hijk} \quad \textit{for some function } F.$$

The relations (4.6) and (4.7) correspond to relation (f) of §1, (4.9) - to relation (h), (4.10) - to (i) and (4.11) - to (j).

Proof. In view of (2.8) and (2.13), we have

$$\alpha_{rt} C^{t}_{ijk} = -\frac{1}{2} a_t a^t C_{rijk}\ ,$$

so that, by virtue of (3.2) and (4.3), we find

(4.12) $$\bar{R}_{it}\bar{C}^{t}_{jkl} = R_{it} C^{t}_{jkl} + \left\{\alpha^{t}_{t} - \frac{n-2}{2} a_t a^t\right\} C_{ijkl}\ .$$

Using (4.4), we get

$$\bar{R}_{it}\bar{C}^{t}_{jkl} = \Pi_{it} C^{t}_{jkl} - \frac{n-2}{2} a_t a^t C_{ijkl}\ .$$

Now, (4.6) follows from Theorem 4 and the relation $\bar{R}_{it}\bar{C}^{t}_{jkl} = 0$ (i.e. relation (f) of §1).

We can prove (4.7) in a similar manner. In fact, transvecting (4.2) with g^{ij}, we find

$$\alpha^{t}_{t} = \nabla_t a^t + \frac{n-2}{2} a_t a^t\ .$$

Substituting this into (4.12), we get

$$\bar{R}_{it}\bar{C}^{t}_{jkl} = R_{it} C^{t}_{jkl} + \nabla_t a^t C_{ijkl}\ .$$

Relation (4.8) is an immediate consequence of (4.7) and (2.3). The relations (4.9), (4.10) and (4.11) are consequences of Theorem 4, (4.5) and (h), (i) and (j) of §1.

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Rezime

NEKE TEOREME O KONFORMNO KVAZI-REKURENTNIM MNOGOSTRUKOSTIMA

Konformno kvazi-rekurentna mnogostukost je n-dimenziona ($n > 3$) Rimanova mnogostrukost čiji tenzor konformne krivine C_{hijk} zadovoljava uslov (1.2). Dokazano je da se takva mnogostrukost, ukoliko je a_i gradijentno vektorsko polje, može konformno preslikati na konformno simetričnu mnogostrukost (tj. na mnogostrukost koja zadovoljava uslov (1.3)). Koristeći tu činjenicu, dokazano je da se mnoge osobine konformno simetričnih mnogostrukosti mogu uopštiti tako da važe za one mnogostrukosti koje su konformno kvazi-rekurentne (a kod kojih je a_i gradijentno vektorsko polje). Takođe su dokazane i neke osobine opštih konformno kvazi-rekurentnih mnogostrukosti.

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