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A NOTE ABOUT A COVARIANT DERIVATIVE OF A HARMONIC VECTOR FIELD IN A RIEMANNIAN MANIFOLD

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Abstract

Betti numbers are related to the topology of the manifold. The one-dimensional Betti number is equal to the number of linearly independent harmonic vector fields on the manifold. Every harmonic vector field is a gradient vector field. Using that fact, we are getting some results about covariant derivative of a harmonic vector field.

Introduction

Let us consider a transformation of the metric tensor on an n-dimensional Riemannian space M

$$\bar{g}_{ik} = e^{2\sigma} g_{ik}. \tag{0.1}$$

Then, the coefficients of their Riemannian connections are related

$$\overline{\left\{ {i \atop jk} \right\}} = \left\{ {i \atop jk} \right\} + \pi_j \delta^i_k + \pi_k \delta^i_j - \pi^i g_{jk}, \tag{0.2}$$

where $\overline{\{\ \}}$ are coefficients of the connection attached to metric $\bar{g}$ and $\{\ \}$ - coefficients of connection attached to the metric g. π_j is a gradient vector field and $\pi_j = \dfrac{\partial\sigma}{\partial x^j}$.

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Since this transformation is an "angle preserving" transformation, we call it a conformal transformation.

Under a conformal transformation of the metric, the curvature tensor will be transformed into

(0.3) $$\bar{R}^{i}_{jkl} = R^{i}_{jkl} + \delta^{i}_{k}\psi_{lj} - \delta^{i}_{l}\psi_{kj} + g_{jl}\psi^{i}_{k} - g_{jk}\psi^{i}_{l},$$

where

$$\psi_{lj} = \nabla_{l}\pi_{j} - \pi_{l}\pi_{j} + \frac{1}{2}\pi_{s}\pi^{s}g_{jl} \; ; \quad \psi^{j}_{l} = g^{jk}\psi_{lk} \; ; \quad \pi^{a} = g^{ab}\pi_{b} \; ,$$

but the tensor

(0.4) $$\bar{C}^{i}_{jkl} = \bar{R}^{i}_{jkl} + \frac{1}{n-2}(\delta^{i}_{k}\bar{R}_{jl} - \delta^{i}_{l}\bar{R}_{jk} + g_{jl}\bar{R}^{i}_{k} - g_{jk}\bar{R}^{i}_{l}) - - \frac{\bar{R}}{(n-1)(n-2)}(\delta^{i}_{k}g_{jl} - \delta^{i}_{l}g_{jk})$$

will be invariant.

Conversely, the tensor (0.4) will be an invariant of a transformation (0.2) if and only if π_{i} is a gradient vector field.

If the vector π_{i} in expression (0.2) satisfies the condition

(0.5) $$\nabla_{k}\pi_{l} = \pi_{k}\pi_{l} + \rho\, g_{lk}$$

(ρ is an arbitrary function), the invariant tensor of such a transformation will be

(0.6) $$\bar{Z}^{i}_{jkl} = \bar{R}^{i}_{jkl} + \frac{R}{n(n-1)}(\delta^{i}_{k}g_{jl} - \delta^{i}_{l}g_{jk}) \; .$$

Condition (0.5) is called a concircularity condition. Tensor (0.4) is called the conformal curvature tensor. Tensor (0.6) is called the concircular curvature tensor, because transformation (0.2) under condition (0.5) sends a geodesic circle into a geodesic circle. The symbols ∇ and $\bar{\nabla}$ denote a covariant differentiation with respect to connections $\{\ \}$ and $\overline{\{\ \}}$.

A vector field is called a harmonic vector field if it satisfies

(0.7) $$\nabla_{k}v_{l} - \nabla_{l}v_{k} = 0$$

and

(0.8) $$\nabla_{k}v^{k} = 0 \; ; \qquad v^{k} = g^{ka}v_{a} .$$

It is well known that the number of linearly independent harmonic vector fields in the manifold is equal to its one-dimensional Betti number. If the one-dimensional Betti number of the manifold is non-zero, then its Ricci curvature form cannot be a positive definite. If the Ricci curvature form in the direction of harmonic vector field vanishes, such a vector field is a parallel vector field.

1. A concircular transformation of the Riemannian connection with a harmonic generator

Suppose that the Riemannian manifold M has non-zero one-dimensional Betti number and the vector field π_i is one of its harmonic vector fields. Then, it is a gradient vector field, and for the curvature tensor of the connection (0.2), we will get the relation

$$\overline{R}_{ijkl} = R_{ijkl} + g_{ik}\psi_{lj} - g_{il}\psi_{kj} + g_{jl}\psi_{ki} - g_{jk}\psi_{li} \tag{1.1}$$

which is, in fact, relation (0.3) after lowering the superscript l. Contracting (1.1) by g^{il}, we get

$$\overline{R}_{jk} = R_{jk} + (2-n)\psi_{kj} - g_{jk}\psi_{il}g^{il}. \tag{1.2}$$

Transvecting (1.2) by g^{jk} we, get

$$\overline{R} = r + 2(1-n)\psi_k{}^k. \tag{1.3}$$

From (1.1), (1.2) and (1.3) there is the invariance of the tensor (0.4).

Now, the vector π_i is harmonic and

$$\psi_k{}^k = g^{jk}\psi_{kj} = \nabla_s\pi^s - \pi_s\pi^s + \frac{1}{2}n\pi_s\pi^s = \frac{n-2}{2}\pi_s\pi^s. \tag{1.4}$$

Then

$$\overline{R} = R - (n-1)(n-2)\pi_s\pi^s \tag{1.5}$$

and

$$\overline{R}_{jk} = R_{jk} - (n-2)(\nabla_k\pi_j - \pi_k\pi_j + \pi_s\pi^s g_{jk}) \tag{1.6}$$

or

$$\overline{R}_{jk} = R_{jk} - (n-2)\psi_{kj} - \frac{n-2}{2}\pi_s\pi^s g_{jk}.$$

Suppose, now, that the vector π_i satisfies the concircularity condition (0.5). Then, (1.7) $$\psi_{kj} = (\rho + \frac{1}{2}\pi_s\pi^s)g_{jk} ,$$

where ρ is the arbitrary function from the concircularity condition (0.5). Then,

(1.8) $$\bar{R}_{ijkl} = R_{ijkl} + (2\rho + \pi_s\pi^s)(g_{ik}g_{lj} - g_{il}g_{kj})$$

(1.9) $$\bar{R}_{jk} = R_{jk} - (n-1)(2\rho + \pi_s\pi^s)g_{jk}$$

and

(1.10) $$\bar{R} = R - n(n-1)(2\rho + \pi_s\pi^s) .$$

As we have supposed earlier, the vector π_i is a harmonic vector field, and according to (0.8) and (0.5)

$$\rho = -\frac{\pi_s\pi^s}{n}$$

and

$$2\rho + \pi_s\pi^s = \frac{n-2}{n}\pi_s\pi^s.$$

Then, (1.8) can be transformed into

(1.8') $$\bar{R}_{ijkl} = R_{ijkl} + \frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{jk}) ,$$

(1.9) can be transformed into

(1.9') $$\bar{R}_{jk} = R_{jk} - \frac{(n-1)(n-2)}{n}\pi_s\pi^s g_{jk} ,$$

and (1.10) can be transformed into

(1.10') $$\bar{R} = R - (n-1)(n-2)\pi_s\pi^s .$$

As the generator of connection (0.2) is a gradient and satisfies the concircularity condition, then the concircular curvature tensor of connection (0.2) has to be equal to the concircular curvature tensor of the Levi-Civita connection.

Using the formulae (1.8'), (1.9') and (1.10'), we can express the covariant components of the concircular curvature tensor Z

$$\bar{Z}_{ijkl} = R_{ijkl} + \frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{kj}) - \frac{R}{n(n-1)}(g_{ik}g_{lj} - g_{il}g_{kj})$$

$$+ \frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{kj}) = Z_{ijkl} + 2\,\frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{kj})$$

As the concircular curvature tensor is invariant under concircular transformations, we get

$$\frac{n-2}{n}\pi_s\pi^s(g_{ik}g_{lj} - g_{il}g_{kj})$$

or, if we presume dim $M > 2$,

(1.11) $$\pi_s\pi^s = 0 .$$

So, we have proved

Theorem 1. *If the one-dimensional Betti number of the differentiable manifold M of a positive definite metric is non-zero, every harmonic vector field can serve as a generator of a connection of type* (0.2), *but none of these transformations is concircular.*

Corollary 1. *On an n-dimensional differentiable manifold M with a positive definite metric and a positive one-dimensional Betti number* B_1, *there does not exist any harmonic vector field which satisfies condition* (0.5).

Corollary 2. *If a harmonic vector field on a differentiable manifold M satisfies condition* (0.5), *its metric form cannot be a positive definite. Such a space can be an Einstein space if and only if such a harmonic vector field is a null-parallel vector field.*

2. Properties of a semi-symmetric metric connection with a harmonic generator

For a semi-symmetric metric connection

$$\Lambda^i_{jk} = \left\{ {i \atop jk} \right\} + \pi_j\delta^i_k - \pi^i g_{jk}$$

there hold

Proposition 1. *a) The conformal curvature tensor of connection* (2.1) *is equal to the conformal curvature tensor of the Levi-Civita connection.*

b) If the vector π_i *satisfies condition* (0.5), *the projective curvature tensor of connection* (2.1) *is equal to the projective curvature tensor of the Levi-Civita connection and the concircular curvature tensor of connection* (2.1) *is equal to the concircular curvature tensor of the Levi-Civita connection.*

Since, according to Corollary 1, no harmonic vector field can satisfy condition (0.5), we have

Corollary 3. *On a differentiable manifold M, with a positive definite metric form, if the vector π_i in expression (2.1) is harmonic, then connection (2.1) has neither geodesic lines nor geodesic circles in common with the Levi-Civita connection.*

3. Covariant derivative of a harmonic vector field

We have made it clear in 1. that the covariant derivative of a harmonic vector field on a differentiable manifold cannot be of the form (0.5). Now, we want to find, if it is possible, a totally general form of an expression which could represent a covariant derivative of a harmonic vector field.

Now, we know that if the vector π_i is harmonic, the concircular curvature tensor of connection (0.3) is not equal to the concircular curvature tensor of the Levi-Civita connection. We shall consider the tensor

$$D_{ijkl} = \bar{Z}_{ijkl} - Z_{ijkl} . \tag{3.1}$$

Transvecting (3.1) by g^{il}, we get

$$g^{il} D_{ijkl} = D_{jk} = \bar{R}_{jk} - R_{jk} - \frac{\bar{R}-R}{n} g_{jk} \tag{3.2}$$

Using the formulae (1.5) and (1.6), we can easily get

$$D_{jk} = -(n-2)(\nabla_k \pi_j - \pi_k \pi_j + \pi_s \pi^s) g_{jk} + \frac{(n-1)(n-2)}{n} \pi_s \pi^s g_{jk}$$

And finally

$$\nabla_k \pi_j = \pi_k \pi_j - \frac{1}{n} \pi_s \pi^s g_{jk} - \frac{1}{n-2} D_{jk} \tag{3.3}$$

We can get even more properties. For example,

$$D_s^{\ s} = \bar{R}_s^s - R_s^s - (\bar{R}-R) = 0$$

Finally, we have the next

Proposition 2. *A vector field π_i is harmonic if and only if*

$$g^{jk} \nabla_k \nabla_j \pi_i = R_j^i \pi^j \tag{3.4}$$

Now, we shall differentiate (3.3) one more time:

$$\nabla_i\nabla_k\pi_j = \pi_k\nabla_i\pi_j + \pi_j\nabla_i\pi_k - \frac{1}{n}g_{jk}(\pi_s\nabla_i\pi^s + \pi^s\nabla_i\pi_s) - \frac{1}{n-2}\nabla_i D_{jk} =$$

$$\pi_k\pi_i\pi_j - \frac{1}{n}\pi_k\pi g_{ij} - \frac{1}{n-2}\pi_k D_{ij} + \pi_j\pi_k\pi_i - \frac{1}{n}\pi\pi_j g_{ik} - \frac{1}{n-2}\pi_j D_{ik} -$$

$$- \frac{1}{n}g_{jk}(\pi_s\pi_i\pi^s - \frac{1}{n}\pi_i\pi - \frac{1}{n-2}\pi_s D^s_i + \pi_s\pi_i\pi_s - \frac{1}{n}\pi\pi_i - \frac{1}{n-2}\pi^3 D_{is}) -$$

$$- \frac{1}{n-2}\nabla_i D_{jk} =$$

$$= 2\pi_k\pi_i\pi_j - \frac{1}{n}\pi_k\pi g_{ij} - \frac{1}{n}\pi\pi_j g_{ik} - \frac{2}{n}\pi\pi_i g_{jk} + \frac{2}{n^2}\pi\pi_i g_{ik}$$

$$+ \frac{2}{n(n-2)}\pi^s D_{is}g_{jk} - \frac{1}{n-2}\pi_k D_{ij} - \frac{1}{n-2}\pi_j D_{ik} - \frac{1}{n-2}\overset{0}{\nabla_i} D_{jk}$$

After transvecting by g^{ik}, we get

$$\frac{(n-1)(n-2)}{n^2}\pi\pi_i - \frac{1}{n}\pi_k D^k_i - \frac{1}{n-2}g^{jk}\nabla_i D_{jk} = R_{ij}\pi^i$$

Transvecting again by π^i, we get

$$\text{(3.5)} \qquad \frac{(n-1)(n-2)}{n^2}\pi^2 - \frac{1}{n}D_{ab}\pi^a\pi^b - \frac{1}{n-2}\pi^j g^{ik}\nabla_i D_{jk} = R_{ij}\pi^i\pi^j$$

We have just proved

Theorem 2. On a Riemannian manifold with an indefinite Ricci form and a positive definite metric, the covariant derivative of a harmonic vector field is of the form (3.3). D_{jk} is a non-zero symmetric tensor field which is not proportional to the metric tensor and wich satisfies

(1) $D_s^{\ s} = 0$

(2) (3.5), where π denotes $\pi_s\pi^s$

(3) $\frac{(n-1)(n-2)}{n^2}\pi^2 - \frac{1}{n}D_{ab}\pi^a\pi^b - \frac{1}{n-2}\pi^j g^{ik}\nabla_i D_{jk} < 0$

(4) $g^{ik}(\nabla_i D_{jk} - \nabla_k D_{ji}) = 0.$

References

1. **L.P. Eisenhart:** An Introduction to Differential Geometry. Princeton University Press 1947.
2. **W. Ślósarska:** On some invariants of Riemannian manifolds admitting a concircularly semi-symmetric metric connection Demonstration Mathematica XVII, 251-257 (1984)
3. **K. Yano and S. Bochner:** Curvature and Betti numbers. Princeton 1953.

Rezime

NOTA O KOVARIJANTNIM IZVODIMA HARMONIJSKOG VEKTORSKOG POLJA
U RIMANOVOJ MNOGOSTRUKOSTI

U radu je izveden najopštiji oblik kovarijantnog izvoda harmonijskog vektorskog polja. Dokazano je da harmonijski vektor ne može zadovoljiti uslov koncirkularnosti ako je metrika pozitivno definitna. Ako je metrika indefinitna, potprostor generisan harmonijskim vektorima je Ajnštajnov ako i samo ako je svaki harmonijski vektor nula-paralelno vektorsko polje.

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