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ON UNAVOIDABLE SUBGRAPHS OF STRONG TOURNAMENTS

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Abstract

If $C(n,i)$ denotes a simple n-cycle $v_1 \to v_2 \to \dots \to v_n \to v_1$ with an additional arc $v_j v_{j+i-1}$ for some $j \in \{1,2,\dots,n\}$ and $3 \le i \le n-1$ it is proved that every strong n-tournament T_n contains copies of $C(n, \lceil (n+2)/2 \rceil)$ and $C(n,n-2)$ for each n $(n \ge 4)$.

The terminology and notation is that of [1] except as noted.

T_n denotes an arbitrary n-tournament. $C(n,i)$ denotes a simple cycle $v_1 \to v_2 \to \dots \to v_n \to v_1$ with an additional arc $v_j v_{j+i-1}$ for some $j \in \{1,2,\dots,n\}$ and $3 \le i \le n-1$. $T_n - v$ denotes the subtournament of T_n obtained by deleting the vertex v and all incident arcs. For vertices u and $v, u \to v$ will be used to denote the phrase "u dominates v" and an arc from u to v in T_n. For two disjoint subsets A and B of $V(T_n)$, $A \to B$ denotes that every vertex of A dominates every vertex of B.

The following theorem will be of use.

Theorem 1. ([2]) *Every strong n-tournament T_n contains a copy of $C(n,i)$ for each n $(n \ge 4)$ and i, $3 \le i \le (n+2)/2$.*

This paper will present some results on digraphs $C(n,i)$ for those i's not covered by Theorem 1.

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Theorem 2. *Every strong n-tournament T_n contains a copy $C(n, [(n+2)/2])$ for each n ($n \geq 4$).*

Proof. For n even or $n \leq 5$ the statement follows immediately by Theorem 1. So, we may assume that $n = 2k + 1$ $(k \geq 3)$ and look for $C(2k+1,\ k+2)$ in strong $(2k+1)$-tournaments. We shall suppose that there is a strong $(2k+1)$-tournament T_{2k+1} containing no copy of $C(2k+1,\ k+2)$ and show that it leads to a contradiction.

Let T_n, $n=2k+1$ $(k \geq 3)$ be a strong tournament which contains no copy of $C(2k+1,\ k+2)$. Let v be a vertex of T_n such that the tournament T_n-v is strong. (For the existence of such a vertex see [1] p.6.) By Theorem 1 the tournament T_n-v contains a copy of $C(2k, k+1)$. Label vertices of T_n-v so that

$$\underline{v_1} \to v_2 \to \dots\ v_k \to \underline{v_{k+1}} \to \dots \to v_{2k} \to v_1$$

is a copy of $C(2k,\ k+1)$. (The underlined pair of vertices denotes that the first one dominates the second.) The proof falls in to three cases.

Case 1. $v_1 \to v$. This implies

(1) $$\{v_2, v_3, \dots, v_{k+1}\} \to v\ .$$

Otherwise, if $v \to v_j$ for some $j \in \{2, 3, \dots, k+1\}$, v can be inserted in the path $v_1 \to v_2 \to \dots \to v_{k+1}$ obtaining a copy of $C(2k+1,\ k+2)$ given by $\underline{v_1} \to \dots \to v \to \dots\ v_{k+1} \to \underline{v_{k+2}} \to \dots \to v_k \to v_1$. So, (1) holds and as T_n-v is strong there is at least one vertex v_i, $i \in \{k+2, k+3, \dots, 2k\}$ such that $v \to v_i$. Let i_0 be the minimal of these i's i.e. $v_{i_0-1} \to v$ and $v \to v_{i_0}$. Then,

$$\underline{v_{i_0-k-1}} \to v_{i_0-k} \to \dots \to v_{i_0-1} \to \underline{v} \to v_{i_0} \to v_{i_0+1} \to \dots \to$$

$$\to v_{i_0-k-2} \to v_{i_0-k-1}$$

is a copy of $C(2k+1,\ k+2)$ as $1 \leq i_0-k-1 \leq k-1$ and by (1) $v_{i_0-k-1} \to v$. Case 1 is settled.

Case 2. $v \to v_{k+1}$. Before discussing this case observe the following property of digraphs $C(n, i)$. Reversing all the arcs of $C(n, i)$ results in a digraph isomorphic to $C(n, i)$. Thus, after reversing all the arcs of T_n we get the tournament T_n which satysfies conditions of case 1 and contains a copy of $C(2k+1,\ k+2)$. Reversing now all the arcs of T_n we get the former tournament T_n and $C(2k+1, k+2)$ in it. In fact, case 2 is the dual of case 1.

Case 3. $v \longrightarrow v_1$ and $v_{k+1} \longrightarrow v$. We claim that now there exists $i_0, k+2 \le i_0 \le 2k$ such that

(2) $$v \longrightarrow \{v_{i_0+1}, v_{i_0+2}, \dots, v_{i_0+k}\}$$

(3) $$\{v_{i_0+k+1}, v_{i_0+k+2}, \dots, v_{i_0}\} \mapsto v$$

(All incides are reduced modulo $2k$.)

First notice if $v \longrightarrow v_j$ for some $j \in \{1,2,\dots,k\}$, then $v \longrightarrow \{v_1, v_2, \dots, v_j\}$. Indeed, if $v_m \longrightarrow v$ for some $m \in \{2,3,\dots,j-1\}$ then v can be inserted in the path $v_1 \longrightarrow v_2 \longrightarrow \dots v_j$ producing a copy of $C(2k+1,k+2)$ given by $\underline{v_1} \longrightarrow \dots \longrightarrow v \longrightarrow \dots \longrightarrow v_j \longrightarrow v_{j+1} \longrightarrow \dots \longrightarrow \underline{v_{k+1}} \longrightarrow v_{k+2} \longrightarrow \dots \longrightarrow v_{2k} \longrightarrow v_1$.

It follows that there exists j_0, $2 \le j_0 \le k$ such that

$$v \longrightarrow \{v_1, v_2, \dots, v_{j_0}\}$$
$$\{v_{j_0+1}, v_{j_0+2}, \dots, v_{k+1}\} \longrightarrow v\ .$$

Further, denote by i_0 the maximum of $k+1$, $k+2, \dots, 2k$, so that

$$\{v_{k+1}, v_{k+2}, \dots, v_{i_0}\} \longrightarrow v\ .$$

We shall show that $i_0 - j_0 \ge k$. (All computations will be done modulo $2k$.) Indeed, if $i_0 - j_0 < k$, then $\underline{v} \longrightarrow v_{i_0+1} \longrightarrow \dots \longrightarrow \underline{v_{i_0+k+1}} \longrightarrow v_{i_0+k+2} \longrightarrow \dots \longrightarrow v_{i_0} \longrightarrow v$ is a copy of $C(2k+1,\ k+2)$. (Notice that $v \longrightarrow v_{i_0+k+1}$ as $1 \le i_0+k+1 \le j_0$.) Thus, $i_0 - j_0 \ge k$.

Using the same argument we derive that there exists I_0, $k+2 \le I_0 \le 2k$ such that $j_0 - I_0 + \ge k$ and $v \longrightarrow \{v_1, v_{I_0+1}, \dots, v_{j_0}\}$. Since $|V(T_n - v)| = 2k$, it follows that

$$i_0 - j_0 \equiv j_0 - I_0 + 1 \equiv k \pmod{2k}$$

i.e. $I_0 = i_0 + 1$. Obviously, this implies (2) and (3).

Relabel now the vertex set of T_n as follows. The vertex v_{i_0+1} is labelled w_1, v_{i_0+2} is labelled $w_2, \dots, v_{i_0}$ is labelled w_{2k} and v is labelled w_{2k+1}. According to this labelling and having in mind (2) and (3), we find that

(4) $$w_1 \longrightarrow w_2 \longrightarrow \dots \longrightarrow w_{2k} \longrightarrow w_{2k+1} \longrightarrow w_1$$

is a Hamiltonian cycle of T_n where

(5)
$$w_{2k+1} \to \{w_1, w_2, \dots, w_k\}$$
$$\{w_{k+1}, w_{k+2}, \dots, w_{2k}\} \to w_{2k+1}$$

This implies that

(6)
$$w_i \to w_{i+k}$$

holds for each $i \in \{1,2,\dots,2k+1\}$. (All computations which follow will be done modulo $2k+1$.) For $i=k+1$ and $i=2k+1$ (6) follows by (5). Suppose that $w_{i+k} \to w_i$ for some $i \in \{1,2,\dots,k,k+2,k+3,\dots,2k\}$. Then, $\underline{w_{i+k}} \to w_{i+k+1} \to \dots \dots \to \underline{w_i} \to w_{i+1} \to \dots \to w_{i+k-1} \to w_{i+k}$ is a copy of $C(2k+1, k+2)$ in T_n. This contradiction proves (6).

Consider now a copy of $C(2k, k+1)$ in $T_n - w_{2k}$ given by $\underline{w_{2k+1}} \to w_1 \to \to w_2 \to \dots \to \underline{w_k} \to w_{k+1} \to \dots \to w_{2k-1} \to w_{2k+1}$ ($w_{2k+1} \to w_k$ by (4).) As, by (4) and (6), $w_{2k} \to w_{2k+1}$, $w_k \to w_{2k}$, $w_{2k} \to w_{k-1}$ we conclude, using the same arguments as for (5), that

$$w_{2k} \to \{w_{2k+1}, w_1, w_2, \dots, w_{k-1}\}$$
$$\{w_k, w_{k+1}, \dots, w_{2k-1}\} \to w_{2k}$$

Continuing this procedure we obtain that

(7)
$$w_i \to \{w_{i+1}, w_{i+2}, \dots, w_{i+k}\}$$
$$\{w_{i+k+1}, w_{i+k+2}, \dots, w_{i-1}\} \to w_i$$

holds for each $i \in \{1,2,\dots,2k+1\}$. So, T_n is the regular tournament with an arc set given by (7). But such a tournament contains a copy of $C(2k+1, k+2)$ given by

$$\underline{w_1} \to w_3 \to w_5 \to \dots \to w_{2k+1} \to \underline{w_2} \to w_4 \to \dots w_{2k} \to w_1$$

This contradiction settles case 3 and completes the proof of the theorem.

Theorem 3. *Every strong n-tournament T_n contains a copy of $C(n, n-2)$ for each $n (n \geq 4)$.*

Proof. By induction on n. For $n \leq 7$ the statement follows by Theorem 1. So, assume that $n \geq 8$. Let T_{n+1} $(n \geq 7)$ be an arbitrary strong $(n+1)$-tournament and let v be a vertex of T_{n+1} such that $T_{n+1} - v$ is strong too. By the induction hypothesis, $T_{n+1} - v$ contains a copy of $C(n, n-2)$ given by

(8) $$\underline{v_1} \to v_2 \to \dots \to \underline{v_{n-2}} \to v_{n-1} \to v_n \to v_1$$

Now we suppose that T_n does not contain a copy of $C(n+1,n-1)$ and show that it leads to a contradiction.

We shall consider three characteristic cases.

Case 1. $v_1 \to v$. Applying the same argument as in case 1 of Theorem 2 we obtain that

(9) $$\{v_2, v_3, \dots, v_{n-2}\} \to v$$

Also, since T_{n+1} is strong, some of the vertices v_{n-1} and v_n dominate v. Let i be the maximum of $n-1$ and n so that $v_i \to v$. Combining this with (7) and (8), we get a copy of $C(n+1,n-1)$ given by

$$\underline{v} \to v_{i+1} \to v_{i+2} \to \dots \to \underline{v_{i-2}} \to v_{i-1} \to v_i \to v$$

(Observe that $1 < i-2 \le n-2$ and $v \to v_{i-2}$) This contradiction settles case 1.

Case 2. $v \to v_{n-2}$. This case is the dual of case 1. Compare this with case 2 of Theorem 2.

Case 3. $v \to v_1$ and $v_{n-2} \to v$. As for the arcs connecting the vertex v with vetrices v_{n-1} and v_n, there are the four possible cases:

a) $v \to \{v_{n-1}, v_n\}$

b) $v \to v_{n-1}$, $v_n \to v$

c) $\{v_{n-1}, v_n\} \to v$

d) $v_{n-1} \to v$, $v \to v_n$

It is easy to see that (*a*) and (*c*) and also (*b*) and (*d*) are dual. So, we shall examine (*a*) and (*b*) only.

a) $v \to \{v_{n-1}, v_n\}$. First notice that $v_{n-4} \to v$. (If on the contrary, $v \to v_{n-4}$, then there is a copy of $C(n+1,n-1)$ given by $\underline{v} \to v_{n-1} \to v_n \to v_1 \to v_2 \to \dots \to \underline{v_{n-4}} \to v_{n-3} \to v_{n-2} \to v$.) This implies

(10) $$v_{n-3} \to v$$

Otherwise, we obtain a copy of $C(n+1,n-1)$ given by $\underline{v_1} \to v_2 \to \dots \to v_{n-4} \to$ $\to v \to v_{n-3} \to \underline{v_{n-2}} \to v_{n-1} \to v_n \to v_1$. Furthermore,

(11) $$v_{n-1} \to v_2$$

In view of a copy of $C(n+1,n-1)$ given by $\underline{v_2} \to v_3 \to \dots \to v_{n-2} \to v \to$ $\to \underline{v_{n-1}} \to v_n \to v_1 \to v_2$.

Finally, using (10) and (11) we still obtain a copy of $C(n+1,n-1)$ given by

$$\underline{v_{n-1}} \to v_2 \to v_3 \to \dots v_{n-3} \to v \to v_n \to v_1 \to v_{n-2} \to v_{n-1} .$$

This contradiction settles (*a*).

(*b*) $v \to v_{n-1}$, $v_n \to v$. By the same argument as in (*a*) (10) and (11) holds. Furthermore,

(12) $$v_n \to v$$

because of the eventual copy of $C(n+1,n-1)$ given by $\underline{v_2} \to v_3 \to \dots \to \underline{v_n} \to$ $\to v \to v_1 \to v_2$. But, now, using (11) and (12) we obtain a copy of $C(n+1,n-1)$ given by

$$\underline{v_n} \to v_2 \to v_3 \to \dots \to v_{n-3} \to v \to \underline{v_1} \to v_{n-2} \to v_{n-1} \to v_n$$

contradicting the assumption.

The proof of the theorem is completed.

This paper and [2] partially confirm following, may be true,

Conjecture. *Every strong n-tournament T_n contains a copy of $C(n,i)$ for each $n(n \ge 4)$ and $i(3 \le i < n-1)$.*

References

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Rezime

NEIZBEŽNI PODGRAFOVI JAKO POVEZANIH TURNIRA

U radu se pokazuje da svaki jako povezan turnir sa n čvorova sadrži kao podgraf $C(n,i)$, gde je $C(n,i)$ orjentisana kontura $v_1 \to v_2 \to \dots \to v_n \to v_1$ sa jednom dodatnim lukom $v_j v_{j+i-1}$, $j \in \{1,2,\dots,n\}$, $i = \lceil (n+2)/2 \rceil$, $i = n-2$, $n \ge 4$.

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