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GENERALIZED PSEUDO-BOOLEAN FUNCTIONAL EQUATIONS OF THE THIRD ORDER

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Abstract

Necessary and sufficient conditions for existence of solutions to the third order pseudo-Boolean functional equation are given as well as the solutions in explicit form.

Introduction

Let $(P,+,\cdot)$ be a commutative ring with identity element 1 without divisors of zero and let L be a finite set. A generalized pseudo-Boolean function (GPB function) is every mapping f of L^n into P, i.e. $f: L^n \to P$, where L^n is a direct power of L.

The definiton of partial derivatives of GPB functions and some properties of these partial derivatives are given by Gilezan in [4].

Definition 1. *A partial derivative of a GPB function $f: L^n \to P$ with the variables x_i $(i=1,2,\dots,n)$ are GPB functions*

$$\frac{\partial f_a}{\partial x_i} : L^n \longrightarrow P \quad \textit{defined by}$$

$$\frac{\partial f_a}{\partial x_i}(X) = f(x_1,\dots,x_{i-1}, a, x_{i+1},\dots,x_n) - f(x), \quad a \in L \tag{1}$$

$$(1 \le i \le n), \qquad \textit{where} \quad X = (x_1,x_2,\dots,x_n).$$

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A partial derivative of higher order of a GPB function is

$$(1') \qquad \frac{\partial^m f_{a_{i_1}\dots a_{i_m}}}{\partial x_{i_1}\dots\partial x_{i_m}} = \frac{\partial}{\partial x_{i_m}} \dots \left(\frac{\partial}{\partial x_{i_2}}\left(\frac{\partial f_{a_{i_1}}}{\partial x_{i_1}}\right) a_{i_2}\right)\dots\Big)a_{i_m}$$

$$a_{i_j} \in L, \quad j=1,2,\dots,m \qquad (m \ge 1)$$

Directly from definition 1 follows that for every $\alpha, \beta, \in L$, for every $c \in P$ and for every couple of GPB functions f and g the following properties hold

$$\frac{\partial\, c\alpha}{\partial\, x_i}\, 0\,; \quad \frac{\partial\, (cf)\alpha}{\partial\, x_i} = c\,\frac{\partial\, f\alpha}{\partial\, x_i}\,; \quad \frac{\partial (f+g)}{\partial\, x_i} = \frac{\partial\, f\alpha}{\partial\, x_i} + \frac{\partial\, g\alpha}{\partial\, x_i}\,;$$

$$\frac{\partial (f\cdot g)}{\partial\, x_i} = \frac{\partial\, f\alpha}{\partial\, x_i}\cdot g + f\,\frac{\partial\, g\alpha}{\partial\, x_i} + \frac{\partial\, f\alpha}{\partial\, x_i}\cdot\frac{\partial\, g\alpha}{\partial\, x_i}\,;$$

$$\frac{\partial^2 f\alpha\beta}{\partial\, x_i\cdot\partial\, x_j} = \frac{\partial^2 f\beta\alpha}{\partial\, x_j\cdot\partial\, x_i}\,, \qquad i\neq j\,;$$

$$\frac{\partial^m f\alpha^m}{\partial\, x_i^m} = (-1)^{m+1}\,\frac{\partial\, f\alpha}{\partial\, x_i}\,, \quad 1 \le m \le n \quad m \text{ is a natural number}, \ 1 \le i \le n.$$

However, the relation

$$F\Bigg(g_1, g_2, \dots, g_k,\ f,\ \frac{\partial\, f\alpha_1}{\partial\, x_1}, \dots, \frac{\partial\, f\alpha_n}{\partial\, x_n}, \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2}, \dots$$

$$\frac{\partial^2 f\alpha_{n-1}\alpha_n}{\partial x_{n-1}\partial x_n}, \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3}, \dots, \frac{\partial^3 f\alpha_{n-2}\alpha_{n-1}\alpha_n}{\partial x_{n-2}\partial x_{n-1}\partial x_n}, \dots, \frac{\partial^n f\alpha_1\dots\alpha_n}{\partial x_1\dots\partial x_n}\Bigg) = 0$$

$\alpha_1, \dots, \alpha_n \in L$ is a generalized pseudo-Boolean functional equation, where $g_1, g_2, \dots, g_n$ known functions, an unknown function f and some of its partial derivatives take place. Hence, the solution of this functional equation has to be found.

Lemma 1. *A functional equation with an unknown GPB function f*

$$\frac{\partial\, f\alpha_1}{\partial\, x_1} = g(x), \qquad \text{where} \quad \alpha_1 \in L,$$

has a solution if and only if

$$g(x_1,\dots,x_{i-1},\alpha_i,x_{i+1},\dots,x_n) = 0.$$

The solutions f are determined by the following formula

$$f(X) = c(x_1,\dots,x_{i-1},\alpha_i,x_{i+1},\dots,x_n) - g\,(X) \quad \text{or else}$$

$$\int_{\alpha_i} g(X)\,dx_i = c(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) - g(X)$$

where c is an arbitrary function of the variables $x_1,\dots,x_n$.

The proof of this lemma is given in [5].

Let us introduce the following notation

$$(\tilde{x}_i) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \quad \text{and}$$

$$(\tilde{\tilde{\alpha}}_i) = (x_1,\dots,x_{i-1},\alpha_i,x_{i+1},\dots,x_n).$$

Theorem 1. *A system of GPB functional equations*

1.1. $\dfrac{\partial\, f\alpha_i}{\partial\, x_i} = P_i(X) \qquad i=1,2,\dots,n, \qquad X = (x_1,\dots,x_n)$

has a solution if and only if

1.2. $P_i(\tilde{\tilde{\alpha}}_i) = 0 \qquad i=1,2,\dots,n$

1.3. $\dfrac{\partial P_j\alpha_i}{\partial\, x_i} = \dfrac{\partial P_i\alpha_j}{\partial\, x_j}, \qquad i,j = 1,2,\dots,n \qquad i\neq j.$

The solutions f are determined by the following

Formula (they are equivalent to each other)

2.1. $f(X) = c - \sum\limits_{k=1} P_{i_k}\,(\tilde{\tilde{\alpha}}_{i_{k-1}},\ \tilde{\tilde{\alpha}}_{i_{k-1}},\dots,\ \tilde{\tilde{\alpha}}_{i_n}) - P_{i_n}(X)$

where $i_1 i_2 \dots i_n$ are permutations of the set $\{1,2,\dots n\}$ and c is an arbitrary constant from P. The proof of this theorem is given in [5].

Lemma 2. *A system of GPB functional equations*

$$\frac{\partial\, f\alpha_1}{\partial\, x_1} = X\,(x_1,x_2,x_3), \quad \frac{\partial\, f\alpha_2}{\partial\, x_2} = Y\,(x_1,x_2,x_3), \quad \frac{\partial\, f\alpha_3}{\partial\, x_3} = Z\,(x_1,x_2,x_3),$$

has a solution if and only if

(1) $$X(\alpha_1,x_1,x_3) = Y(x_1,\alpha_2,x_3) = Z(x_1,x_2,\alpha_3) = 0$$

(2) $$\frac{\partial X\alpha_2}{\partial x_2} = \frac{\partial Y\alpha_1}{\partial x_1} \text{ and } \frac{\partial X\alpha_3}{\partial x_3} = \frac{\partial Z\alpha_1}{\partial x_1} \text{ and } \frac{\partial Y\alpha_3}{\partial x_3} = \frac{\partial Z\alpha_2}{\partial x_2}$$

All the function solutions are determined by the following formula

(3) $$f(x_1,x_2,x_3) = C - X(x_1,x_2,x_3) - Y(x_1,x_2,x_3) - Z(x_1,x_2,x_3) - \frac{\partial Y\alpha_3}{\partial x_3} - \frac{\partial X\alpha_2}{\partial x_2} - \frac{\partial X\alpha_3}{\partial x_3} - \frac{\partial^2 X\alpha_2\alpha_3}{\partial x_2\,\partial x_3}$$

The proof of this lemma follows from Theorem 1.

Let us consider a functional equation of the following form

(4) $$F:\ a\frac{\partial f\alpha_1}{\partial x_1} + b\frac{\partial f\alpha_2}{\partial x_2} + c\frac{\partial f\alpha_3}{\partial x_3} + d\frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2} + e\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} + g\frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2\partial x_3} + h\frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = s(x_1,x_2,x_3)$$

a,b,c,d,e,g,h and s are GPB functions which mapp $L^3 \to P$.

Theorem 2. *A functional equation (4) has a solution if and only if*

(5)

$$(u)\quad s + \frac{\partial s\alpha_1}{\partial x_1} + \frac{\partial s\alpha_2}{\partial x_2} + \frac{\partial s\alpha_3}{\partial x_3} + \frac{\partial^2 s\alpha_1\alpha_2}{\partial x_1\partial x_2} + \frac{\partial^2 s\alpha_1\alpha_3}{\partial x_1\partial x_3} + \frac{\partial^3 s\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = 0$$

$$(u')\quad (a+b+c-g-e-d+h)\left[a + \frac{\partial a\alpha_2}{\partial x_2} + \frac{\partial a\alpha_3}{\partial x_3} + \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3}\right]\left[b + \frac{\partial b\alpha_1}{\partial x_1} + \frac{\partial b\alpha_3}{\partial x_3} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3}\right]\cdot$$
$$\left[c + \frac{\partial c\alpha_1}{\partial x_1} + \frac{\partial c\alpha_2}{\partial x_2} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2}\right]\left[d + \frac{\partial d\alpha_3}{\partial x_3} - a - \frac{\partial a\alpha_3}{\partial x_3} - b - \frac{\partial b\alpha_3}{\partial x_3}\right]\left[e + \frac{\partial e\alpha_2}{\partial x_2} - a -\right.$$
$$\left.- \frac{\partial a\alpha_2}{\partial x_2} - c - \frac{\partial c\alpha_2}{\partial x_2}\right]\left[g + \frac{\partial g\alpha_1}{\partial x_1} - b - \frac{\partial b\alpha_1}{\partial x_1} - c - \frac{\partial c\alpha_1}{\partial x_1}\right] = 1.$$

If the conditions (5) (u) and (5)(u') are fulfilled, then all the functions f are determined by the formula (3), where, X, Y and Z are given further on the pages 9, 10, 11, respectively.

Proof. If the partial derivatives of the equation (3) are found, i.e.

$$\frac{\partial\ F\alpha_1}{\partial\ x_1},\ \frac{\partial\ F\alpha_2}{\partial\ x_2},\ \frac{\partial\ F\alpha_3}{\partial\ x_3},\ \frac{\partial^2 F\alpha_1\alpha_2}{\partial x_1\partial x_2},\ \frac{\partial^2 F\alpha_1\alpha_3}{\partial x_1\partial x_3},\ \frac{\partial^2 F\alpha_2\alpha_3}{\partial x_2\partial x_3}\ \text{and}\ \frac{\partial^3 F\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3}$$

then the following system of equations is obtained

$$\frac{\partial\ F\alpha_1}{\partial\ x_1},\ \frac{\partial\ F\alpha_2}{\partial\ x_2},\ \frac{\partial\ F\alpha_3}{\partial\ x_3},\ \frac{\partial^2 F\alpha_1\alpha_2}{\partial x_1\partial x_2},\ \frac{\partial^2 F\alpha_1\alpha_3}{\partial x_1\partial x_3},\ \frac{\partial^2 F\alpha_2\alpha_3}{\partial x_2\partial x_3}\ \text{and}\ \frac{\partial^3 F\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3}$$

then the following system of equations is obtained

$$a\frac{\partial\ f\alpha_1}{\partial\ x_1} + b\frac{\partial\ f\alpha_2}{\partial\ x_2} + c\frac{\partial\ f\alpha_3}{\partial\ x_3} + d\frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2} + e\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} + g\frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2\partial x_3} +$$
$$+ g\frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = s$$

$$(6)\quad - a\frac{\partial f\alpha_1}{\partial x_1} + \frac{\partial b\alpha_1}{\partial x_1}\frac{\partial f\alpha_2}{\partial x_2} + \frac{\partial c\alpha_1}{\partial x_1}\frac{\partial f\alpha_3}{\partial x_3} + \left[b + \frac{\partial b\alpha_1}{\partial x_1} - d\right]\frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2} +$$
$$+ \left[c + \frac{\partial c\alpha_1}{\partial x_1} - e\right]\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} + \frac{\partial g\alpha_1}{\partial x_1}\frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2\partial x_3} + \left[g + \frac{\partial g\alpha_1}{\partial x_1} - h\right]\frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = \frac{\partial s\alpha_1}{\partial x_1},$$

$$\frac{\partial a\alpha_2}{\partial x_2}\frac{\partial f\alpha_1}{\partial x_1} - b\frac{\partial f\alpha_2}{\partial x_2} + \frac{\partial c\alpha_2}{\partial x_2}\frac{\partial f\alpha_3}{\partial x_3} + \left[a + \frac{\partial a\alpha_2}{\partial x_2} - d\right]\frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2} + \frac{\partial e\alpha_2}{\partial x_2}\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} +$$
$$+ \left[c + \frac{\partial c\alpha_2}{\partial y} - g\right]\frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2\partial x_3} + \left[e + \frac{\partial e\alpha_2}{\partial x_2} - h\right]\frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = \frac{\partial s\alpha_2}{\partial x_2},$$

$$\frac{\partial a\alpha_3}{\partial x_3}\frac{\partial f\alpha_1}{\partial x_1} + \frac{\partial b\alpha_3}{\partial x_3}\frac{\partial f\alpha_2}{\partial x_2} - c\frac{\partial f\alpha_2}{\partial x_2} + \frac{\partial d\alpha_3}{\partial x_3}\frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2} + \left[a + \frac{\partial a\alpha_3}{\partial x_3} - e\right]\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} +$$
$$+ \left[b + \frac{\partial b\alpha_3}{\partial x_3} - g\right]\frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2\partial x_3} + \left[d + \frac{\partial d\alpha_3}{\partial x_3} - h\right]\frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = \frac{\partial s\alpha_3}{\partial x_3},$$

$$- \frac{\partial a\alpha_2}{\partial x_2}\frac{\partial f\alpha_1}{\partial x_1} - \frac{\partial b\alpha_1}{\partial x_1}\frac{\partial f\alpha_2}{\partial x_2} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2}\frac{\partial f\alpha_3}{\partial x_3} + \left[-a - \frac{\partial a\alpha_2}{\partial x_2} - b - \frac{\partial b\alpha_1}{\partial x_1} + d\right]\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} +$$
$$+ \left[\frac{\partial c\alpha_2}{\partial x_2} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2} - \frac{\partial e\alpha_2}{\partial x_2}\right]\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} + \left[\frac{\partial c\alpha_1}{\partial x_1} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2} - \frac{\partial g\alpha_1}{\partial x_1}\right]\frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} + \qquad (6)$$

$$+ \left(c + \frac{\partial c\alpha_2}{\partial x_2} + \frac{\partial c\alpha_1}{\partial x_1} + \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2} - g - \frac{\partial g\alpha_1}{\partial x_1} + h \right) \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = \frac{\partial^2 s\alpha_1\alpha_2}{\partial x_1\partial x_2} ,$$

$$- \frac{\partial a\alpha_3}{\partial x_3}\frac{\partial f\alpha_1}{\partial x_1} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3}\frac{\partial f\alpha_2}{\partial x_2} - \frac{\partial c\alpha_1}{\partial x_1}\frac{\partial f\alpha_3}{\partial x_3} + \left(\frac{\partial b\alpha_3}{\partial x_3} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3} - \frac{\partial d\alpha_3}{\partial x_3} \right) \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2} +$$

$$\left(-a - \frac{\partial a\alpha_3}{\partial x_3} - c - \frac{\partial c\alpha_1}{\partial x_1} + e \right) \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} + \left(\frac{\partial b\alpha_1}{\partial x_1} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3} - \frac{\partial g\alpha_1}{\partial x_1} \right) \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2\partial x_3} +$$

$$+ \left(b + \frac{\partial b\alpha_3}{\partial x_3} + \frac{\partial b\alpha_1}{\partial x_1} + \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3} - d - \frac{\partial d\alpha_3}{\partial x_3} - g\, \frac{\partial g\alpha_1}{\partial x_1} + h \right) \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = \frac{\partial^2 s\alpha_1\alpha_3}{\partial x_1\partial x_3} ,$$

$$\frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3}\frac{\partial f\alpha_1}{\partial x_1} - \frac{\partial b\alpha_3}{\partial x_3}\frac{\partial f\alpha_2}{\partial x_2} - \frac{\partial c\alpha_2}{\partial x_2}\frac{\partial f\alpha_2}{\partial x_2} + \left(\frac{\partial a\alpha_3}{\partial x_3} + \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3} - \frac{\partial d\alpha_3}{\partial x_3} \right) \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2} +$$

$$+ \left(\frac{\partial a\alpha_2}{\partial x_2} + \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3} - \frac{\partial e\alpha_2}{\partial x_2} \right) \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} + \left(-b - c - \frac{\partial b\alpha_3}{\partial x_3}\frac{\partial c\alpha_2}{\partial x_2} + g \right) \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2\partial x_3} +$$

$$+ \left(a + \frac{\partial a\alpha_3}{\partial x_3} + \frac{\partial a\alpha_2}{\partial x_2} + \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3} - d - \frac{\partial d\alpha_3}{\partial x_3} - e - \frac{\partial e\alpha_1}{\partial x_1} + h \right) \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} = \frac{\partial^3 s\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} ,$$

$$\frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3}\frac{\partial f\alpha_1}{\partial x_1} - \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3}\frac{\partial f\alpha_2}{\partial x_2} - \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2}\frac{\partial f\alpha_3}{\partial x_3} + \left(- \frac{\partial a\alpha_3}{\partial x_3} - \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3} - \right.$$

$$\left. - \frac{\partial b\alpha_3}{\partial x_3} - \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3} + \frac{\partial d\alpha_3}{\partial x_3} \right) \frac{\partial^2 f\alpha_1\alpha_2}{\partial x_1\partial x_2} + \left(- \frac{\partial a\alpha_2}{\partial x_2} - \frac{\partial c\alpha_2}{\partial x_2} - \frac{\partial^2 c\alpha_1\alpha_3}{\partial x_1\partial x_3} + \frac{\partial e\alpha_2}{\partial x_2} - \right.$$

$$\left. - \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3} \right) \frac{\partial^2 f\alpha_1\alpha_3}{\partial x_1\partial x_3} + \left(- \frac{\partial b\alpha_1}{\partial x_1} - \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3} - \frac{\partial c\alpha_1}{\partial x_1} - \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2} + \frac{\partial g\alpha_1}{\partial x_1} \right) \frac{\partial^2 f\alpha_2\alpha_3}{\partial x_2\partial x_3} +$$

$$+e-a-b-c+d+g-h- \frac{\partial a\alpha_3}{\partial x_3} - \frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3} - \frac{\partial a\alpha_2}{\partial x_2} - \frac{\partial b\alpha_3}{\partial x_3} - \frac{\partial b\alpha_1}{\partial x_1} - \frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3} -$$

$$\left. - \frac{\partial c\alpha_2}{\partial x_2} - \frac{\partial c\alpha_1}{\partial x_1} - \frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2} + \frac{\partial d\alpha_3}{\partial x_3} + \frac{\partial e\alpha_2}{\partial x_2} - \frac{\partial g\alpha_1}{\partial x_1} - \frac{\partial^2 g\alpha_1\alpha_3}{\partial x_1\partial x_3} \right) \frac{\partial^3 f\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} =$$

$$= \frac{\partial^3 s\alpha_1\alpha_2\alpha_3}{\partial x_1\partial x_2\partial x_3} . \qquad (6)$$

$$
\Delta' = \begin{bmatrix}
a & b & c & d-a-b & e-a-c & g-b-c & a+b+c-g-e-d-h \\
\text{[illegible]} & b+\frac{\partial b\alpha_1}{\partial x_1} & c+\frac{\partial c\alpha_1}{\partial x_1} & 0 & 0 & g+\frac{\partial g\alpha_1}{\partial x_1}-b-\frac{\partial b\alpha_1}{\partial x_1}-c-\frac{\partial c\alpha_1}{\partial x_1} & 0 \\
a+\frac{\partial a\alpha_2}{\partial x_2} & 0 & c+\frac{\partial c\alpha_2}{\partial x_2} & 0 & e+\frac{\partial e\alpha_2}{\partial x_2}-a-\frac{\partial a\alpha_2}{\partial x_2}-c-\frac{\partial c\alpha_2}{\partial x_2} & 0 & 0 \\
a+\frac{\partial a\alpha_3}{x_3} & b+\frac{\partial b\alpha_3}{\partial x_3} & 0 & -a-\frac{\partial a\ _3}{\partial x_3}-b-\frac{\partial b\alpha_3}{\partial x_3}+d+\frac{\partial d\alpha_3}{\partial x_3} & 0 & 0 & 0 \\
0 & 0 & c+\frac{\partial c\alpha_2}{\partial x_2}+\frac{\partial c\alpha_1}{\partial x_1}+\frac{\partial^2 c\alpha_1\alpha_2}{\partial x_1\partial x_2} & 0 & 0 & 0 & 0 \\
0 & b+\frac{\partial b\alpha_1}{\partial x_1}+\frac{\partial b\alpha_3}{\partial x_3}+\frac{\partial^2 b\alpha_1\alpha_3}{\partial x_1\partial x_3} & 0 & 0 & 0 & 0 & 0 \\
a+\frac{\partial a\alpha_2}{\partial x_2}+\frac{\partial a\alpha_3}{\partial x_3}+\frac{\partial^2 a\alpha_2\alpha_3}{\partial x_2\partial x_3} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$X = \dfrac{\partial[\alpha_1]}{\partial x_1} =$

s	b	c	$d-b-e$	e	$g-b-c$	$h+b-g-d$
$s+\frac{\partial s_{\alpha_1}}{\partial x_1}$	$b+\frac{\partial b_{\alpha_1}}{\partial x_1}$	$c+\frac{\partial c_{\alpha_1}}{\partial x_1}$	0	0	$g+\frac{\partial g_{\alpha_1}}{\partial x_1}-b-\frac{\partial b_{\alpha_1}}{\partial x_1}-c-\frac{\partial c_{\alpha_1}}{\partial x_1}$	0
$s+\frac{\partial s_{\alpha_2}}{\partial x_2}$	0	$c+\frac{\partial c_{\alpha_2}}{\partial x_2}$	$a+\frac{\partial a_{\alpha_2}}{\partial x_2}+c+\frac{\partial c_{\alpha_2}}{\partial x_2}-e-\frac{\partial e_{\alpha_2}}{\partial x_2}$	$e+\frac{\partial e_{\alpha_2}}{\partial x_2}-c-\frac{\partial c_{\alpha_2}}{\partial x_2}$	0	$\frac{\partial e_{\alpha_2}}{\partial x_2}-\frac{\partial c_{\alpha_2}}{\partial x_2}-\frac{\partial a_{\alpha_2}}{\partial x_2}-e-a-c$
$s+\frac{\partial s_{\alpha_3}}{\partial x_3}$	$b+\frac{\partial b_{\alpha_3}}{\partial x_3}$	0	$d+\frac{\partial d_{\alpha_3}}{\partial x_3}-b-\frac{\partial b_{\alpha_3}}{\partial x_3}-a-\frac{\partial a_{\alpha_3}}{\partial x_3}$	$a+\frac{\partial a_{\alpha_3}}{\partial x_3}$	0	0
$s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial^2 s_{\alpha_1\alpha_2}}{\partial x_1\partial x_2}$	0	$c+\frac{\partial c_{\alpha_1}}{\partial x_1}+\frac{\partial c_{\alpha_2}}{\partial x_2}+\frac{\partial^2 c_{\alpha_1\alpha_2}}{\partial x_1\partial x_2}$	0	0	0	0
$s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_1\alpha_3}}{\partial x_1\partial x_3}$	$b+\frac{\partial b_{\alpha_1}}{\partial x_1}+\frac{\partial b_{\alpha_3}}{\partial x_3}+\frac{\partial^2 b_{\alpha_1\alpha_3}}{\partial x_1\partial x_3}$	0	0	0	0	0
$s+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_2\alpha_3}}{\partial x_2\partial x_3}$	0	0	0	$a+\frac{\partial a_{\alpha_2}}{\partial x_2}+\frac{\partial a_{\alpha_3}}{\partial x_3}+\frac{\partial^2 a_{\alpha_2\alpha_3}}{\partial x_2\partial x_3}$	0	0

$Y = \frac{\partial f_{\alpha_2}}{\partial x_2} =$

a	s	c	d-a	e-a-c	g-c-h+e	h+a-d-c
0	$s+\frac{\partial s_{\alpha_1}}{\partial x_1}$	$c+\frac{\partial c_{\alpha_1}}{\partial x_1}$	$b+\frac{\partial b_{\alpha_1}}{\partial x_1}$	0	0	$g+\frac{\partial g_{\alpha_1}}{\partial x_1}-c-\frac{\partial c_{\alpha_1}}{\partial x_1}-b-\frac{\partial b_{\alpha_1}}{\partial x_1}$
$a+\frac{\partial a_{\alpha_2}}{\partial x_2}$	$s+\frac{\partial s_{\alpha_2}}{\partial x_2}$	$c+\frac{\partial c_{\alpha_2}}{\partial x_2}$	0	$e+\frac{\partial e_{\alpha_2}}{\partial x_2}-a-\frac{\partial a_{\alpha_2}}{\partial x_2}-c-\frac{\partial c_{\alpha_2}}{\partial x_2}$	0	0
$a+\frac{\partial a_{\alpha_3}}{\partial x_3}$	$s+\frac{\partial s_{\alpha_3}}{\partial x_3}$	0	$d+\frac{\partial d_{\alpha_3}}{\partial x_3}-a-\frac{\partial a_{\alpha_3}}{\partial x_3}$	0	$a+\frac{\partial a_{\alpha_3}}{\partial x_3}+b+\frac{\partial b_{\alpha_3}}{\partial x_3}-d-\frac{\partial d_{\alpha_3}}{\partial x_3}$	0
0	$s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial^2 s_{\alpha_1\alpha_2}}{\partial x_1 \partial x_2}$	$c+\frac{\partial c_{\alpha_1}}{\partial x_1}+\frac{\partial c_{\alpha_2}}{\partial x_2}+\frac{\partial^2 c_{\alpha_1\alpha_2}}{\partial x_1 \partial x_2}$	0	0	0	0
0	$s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_1\alpha_3}}{\partial x_1 \partial x_3}$	0	$b+\frac{\partial b_{\alpha_1}}{\partial x_1}+\frac{\partial b_{\alpha_3}}{\partial x_3}+\frac{\partial^2 b_{\alpha_1\alpha_3}}{\partial x_1 \partial x_3}$	0	0	0
$a+\frac{\partial a_{\alpha_2}}{\partial x_2}+\frac{\partial a_{\alpha_3}}{\partial x_3}+\frac{\partial^2 a_{\alpha_2\alpha_3}}{\partial x_2 \partial x_3}$	$s+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_2\alpha_3}}{\partial x_2 \partial x_3}$	0	0	0	0	0

$$Z = \frac{\partial \Gamma_{\alpha_3}}{\partial x_3} = \begin{vmatrix} a & b & s & d-a-b & d-b-h & g+h-d & h+b+e-a-d \\ 0 & b+\frac{\partial b_{\alpha_1}}{\partial x_1} & s+\frac{\partial s_{\alpha_1}}{\partial x_1} & 0 & 0 & g+\frac{\partial g_{\alpha_1}}{\partial x_1}-b-\frac{\partial b_{\alpha_1}}{\partial x_1} & c+\frac{\partial c_{\alpha_1}}{\partial x_1} \\ a+\frac{\partial a_{\alpha_2}}{\partial x_2} & 0 & s+\frac{\partial s_{\alpha_2}}{\partial x_2} & 0 & e+\frac{\partial e_{\alpha_2}}{\partial x_2}-a-\frac{\partial a_{\alpha_2}}{\partial x_2} & c+\frac{\partial c_{\alpha_2}}{\partial x_2}+a+\frac{\partial a_{\alpha_2}}{\partial x_2}-e-\frac{\partial e_{\alpha_2}}{\partial x_2} & 0 \\ a+\frac{\partial a_{\alpha_3}}{x_3} & b+\frac{\partial b_{\alpha_3}}{\partial x_3} & s+\frac{\partial s_{\alpha_3}}{\partial x_3} & d+\frac{\partial d_{\alpha_3}}{\partial x_3}-a-\frac{\partial a_{\alpha_3}}{\partial x_3}-b-\frac{\partial b_{\alpha_3}}{\partial x_3} & 0 & 0 & 0 \\ 0 & 0 & s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial^2 s_{\alpha_1\alpha_2}}{\partial x_1 \partial x_2} & 0 & c+\frac{\partial c_{\alpha_1}}{\partial x_1}+\frac{\partial c_{\alpha_2}}{\partial x_2}+\frac{\partial^2 c_{\alpha_1\alpha_2}}{\partial x_1 \partial x_2} & 0 & 0 \\ 0 & b+\frac{\partial b_{\alpha_1}}{\partial x_1}+\frac{\partial b_{\alpha_3}}{\partial x_3}+\frac{\partial^2 b_{\alpha_1\alpha_3}}{\partial x_1 \partial x_3} & s+\frac{\partial s_{\alpha_1}}{\partial x_1}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_1\alpha_3}}{\partial x_1 \partial x_3} & 0 & 0 & 0 & 0 \\ a+\frac{\partial a_{\alpha_2}}{\partial x_2}+\frac{\partial a_{\alpha_3}}{\partial x_3}+\frac{\partial^2 a_{\alpha_2\alpha_3}}{x_2 \, x_3} & 0 & s+\frac{\partial s_{\alpha_2}}{\partial x_2}+\frac{\partial s_{\alpha_3}}{\partial x_3}+\frac{\partial^2 s_{\alpha_2\alpha_3}}{\partial x_2 \partial x_3} & 0 & 0 & 0 & 0 \end{vmatrix}$$

The system (6) has a solution for $\frac{\partial f\alpha_1}{\partial x_1}$, $\frac{\partial f\alpha_2}{\partial x_2}$ and $\frac{\partial f\alpha_3}{\partial x_3}$ if the rank of the augmented matrix of system (6) is 7.

The augmented matrix of system (6) and the following matrix are equivalent.

The rank of the augmented matrix = rank $\Delta' = 7$, if conditions (5)(u) and (5)(u') are fulfilled.

If condition (5)(u) is fulfilled, then the following determinants give the solutions of the partial derivatives, so the final solution of the functional equation (4) has the following form

$$f(x_1,x_2,x_3) = C - X(x_1,x_2,x_3) - Y(x_1,x_2,x_3) - Z(x_1,x_2,x_3) - \frac{\partial Y\alpha_3}{\partial x_3} - \frac{\partial X\alpha_2}{\partial x_2} - \frac{\partial X\alpha_3}{\partial x_3} - \frac{\partial^2 X\alpha_2\alpha_3}{\partial x_2 \partial x_3}.$$

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Rezime

GENERALISANE PSEUDO-BULOVE FUNKCIONALNE JEDNAČINE TREĆEG REDA

U radu su dati potrebni i dovoljni uslovi za postojanje rešenja generalisane pseudo-Bulove funkcionalne jednačine trećeg reda kao i sama njena rešenja u eksplicitnom obliku.

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