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A QUADRATIC SPLINE DIFFERENCE SCHEME FOR A SELF-ADJOINT BOUNDARY VALUE PROBLEM

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Abstract

The exponentially fitted quadratic spline difference scheme for the problem: $-\varepsilon y'' + q(x)y = f(x)$, $0 < x < 1$, $y(0) = \alpha_0$, $y(1) = \alpha_1$ is derived. The scheme has a second order accuracy, under some conditions, on the functions q and f. The numerical results are also given.

1. Introduction

The quadratic spline collocation method ([3], [2]) when applied with a uniform mesh of size h to problem

$$(1)\qquad \begin{cases} Ly = -\varepsilon y'' + q(x)y = f(x), \quad 0 < x < 1, \\ y(0) = \alpha_0, \quad y(1) = \alpha_1 \end{cases}$$

has the condition

$$(2)\qquad \begin{array}{l} h^2 q_{i+1/2}/\varepsilon \le 1, \quad q_{i+1/2} = q(x_i + \frac{h}{2}), \\ h = x_i - x_{i-1}, \quad i = 1(1)n+1, \quad x_0 = 0, \quad x_{n+1} = 1 \end{array}$$

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which leads to spurious oscillation or gross inaccuracies in the approximate solution. In order to avoid this difficulty, we introduce an exponentially fitting factor

(3) $$\sigma_i = q_{i+1/2}h^2\left(1 + \frac{2}{sh^2\rho_i}\right)/8 , \quad \rho_i = h\sqrt{q_{i+1/2}/\varepsilon} ,$$

affecting the second derivative. Because of this the spline collocation method becomes uniformly stable in ε and the corresponding difference scheme is uniformly convergent. The exponentially fitted cubic spline difference scheme for the same problem is derived in [4]. According to [3] the quadratic spline has some adventages over the cubic spline in solving the above problem. The numerical reulsts illustrate this fact.

2. Derivation of the scheme

We shall seek the approximate solution of problem (1) in the form of the quadratic spline $v(x) \in C^1[0,1]$ and on each interval $[x_i, x_{i+1}]$ $v(x)$ has the form

$$v_i(x) = v_i^{(0)} + (x-x_i)v_i^{(1)} + \frac{(x-x_i)^2}{2}v_i^{(2)} .$$

The constants $v_i^{(k)}$, $k=0(1)2$ are obtained from the equations:

(4) $$\sigma_i v_i^{(2)} + q_{i+1/2}\left[v_i^{(0)} + (x-x_i)v_i^{(1)} + (x-x_i)\frac{v_i^{(2)}}{2}\right]_{x=x_i+\frac{h}{2}} = f_{i+1/2}$$

(5) $$v_i^{(0)} = v_{i-1}^{(0)} + hv_{i-1}^{(1)} + \frac{h^2}{2}v_{i-1}^{(2)}$$

(6) $$v_i^{(1)} = v_{i-1}^{(1)} + hv_{i-1}^{(2)}$$

$$v_0 = \alpha_0 , \quad v_{n+1} = \alpha_1$$

From (6) we have $v_{i-1}^{(2)} = \dfrac{v_i^{(1)} - v_{i-1}^{(1)}}{h}$ and, then, from (4) and (5) we have

(7) $$p_i v_{i+1}^{(1)} + s_i v_i^{(1)} = f_{i+1/2} - q_{i+1/2}v_i$$

$$p_i = -\sigma_i + h^2q_{i+1/2}/8 , \quad s_i = \sigma_i + 3h^2q_{i+1/2}/8 .$$

(8) $$v_{i+1}^{(1)} + v_i^{(1)} = 2(v_{i+1} - v_i)/h\ , \quad v_i = v_i^{(0)}.$$

From (7) and (8) we get

(9) $$v_i^{(1)} = \frac{1}{s_i - p_i}\ [(v_{i+1} - v_i)\cdot 2p_i/h + f_{i+1/2}h - q_{i+1/2}hv_i], \quad i=1(1)n,$$

(10) $$v_{i+1}^{(1)} = \frac{1}{p_i - s_i}\ [f_{i+1/2}h - q_{i+1/2}hv_i^{(0)} + 2s_i(v_{i+1} - v_i)/h], \quad i=0(1)n-1.$$

If we replace the subscript i by $i-1$ in (10) and, then, equalize (9) and (10), we obtain the scheme:

(11) $R_h v_i = Q_h f_i$, $\quad i=1(1)n$ where

$$R_h v_i = r^- v_{i-1} + r^c v_i + r^+ v_{i+1}$$

$$Q_h f_i = q^- f_{i-1/2} + q^+ f_{i+1/2}$$

$$r^- = -\frac{1}{h(1 + sh^2\rho_{i-1})}\ , \qquad r^+ = -\frac{1}{h(1 + sh^2\rho_i)}$$

$$r^c = -r^- - r^+ + a_i + a_{i-1}, \qquad a_i = \frac{2sh^2\rho_i}{h(sh^2\rho_i + 1)}$$

$$q^+ = q^+(q_i) = \frac{2sh^2\rho_i}{hq_{i+1/2}(1+sh^2\rho_i)}\ , \qquad q^- = q^+(q_{i-1})$$

3. The truncation error

The truncation error of scheme (11) for an arbitrary smooth function $r(x)$ is defined by

$$\tau_i(r) = R_h r_i - Q_h(Lr)_i$$

Lemma 1. *The truncation error of scheme* (11) *for* $y(x) \in C^3[0,1]$ *has the form:*

$$\tau_i(y) = -(-s_{i-1}\varphi_{i-1,1} + \varphi_{i-1,0})\frac{h}{p_{i-1} - s_{i-1}} +$$

$$+ \frac{h}{s_i - p_i}(-\varphi_{i,1}p_i + \varphi_{i,0})\ , \quad \text{where}$$

$$\varphi_{i,1} = \psi_{i,1} - 2\psi_{i,0}/h, \quad \psi_{i,k} = \frac{(h/2)^{3-k}}{(3-k)!} y'''(\xi_k), \quad x_{i-1} \le \xi_k \le x_i,$$

$$\varphi_{i,0} = (-\sigma_i + q_{i+1/2} \frac{h^2}{8}) \psi_{i,1} - q_{i+1/2} \psi_{i,0} h + + h [(\varepsilon-\sigma_i) y''_{i+1/2} + \sigma_i \psi_{i,2}].$$

Proof.

$\tau_i(y) = R_h(y_i - v_i) = R_h z_i$. The function $z(x)$,

$z(x) = y(x) - v(x) \in C^1[0,1]$, satisfies the following system equations

$$-\sigma_i z_i^{(2)} + q_{i+1/2} (z_i^{(0)} + \frac{h}{2} z_i^{(1)} + \frac{h^2}{8} + \psi_{i,0}) = f_{i+1/2} - (\sigma_i - \varepsilon) y''_i$$

$$z_i^{(0)} = z_{i-1}^{(0)} + h z_{i-1}^{(1)} + \frac{h^2}{2} z_{i-1}^{(2)} + \psi_{i,0}$$

$$z_i^{(1)} = z_{i-1}^{(1)} + h z_{i-1}^{(2)} + \psi_{i,0}$$

By eliminating the constants $v_i^{(1)}$ and $v_i^{(2)}$ from the above equations in the same manner as in the derivation of the scheme, we obtain the statement of Lemma 1.

Lemma 2. *The truncation error of scheme* (11) *for the function* $y(x) \in C^2[0,1]$ has the form:

$$\tau_i(y) = \frac{h^2}{2} r^- y''(\xi_1) + \frac{h^2}{2} r^+ y''(\xi_2) + \varepsilon(q^- y''_{j-1/2} + q^+ y''_{j+1/2}).$$

$$x_{i-1} \le \xi_1 \le x_i \le \xi_2 \le x_{i+1}.$$

Proof. It follows from Taylor developments at x_j.

4. Proof of the uniform convergence

The following lemma from [1] gives us the property of the exact solution which we shall take into account in the proof.

Lemma 3. *Let* $y(x) \in C^4[0,1]$. *Let* $q'(0) = q'(1) = 0$. *Then, the solution of* (1) *has the form*

$$y(x) = u(x) + w(x) + g(x), \quad \textit{where}$$

$$u(x) = a_0 \exp(-x\sqrt{q(0)/\varepsilon})$$

$$w(x) = a_1 \exp(-(1-x)\sqrt{q(1)/\varepsilon}),$$

$$|g^{(i)}(x)| \le M(1 + \varepsilon^{(1-i/2)i}), \quad i=0(1)4,$$

a_0, a_1 are bounded functions of ε independent of x, and M is a constant independent of ε.

Theorem 1. *Let the conditions of Lemma 1 be fulfilled. Let v_j, $j=0(1)n+1$ be the approximation to the solution $y(x)$ of (1) at the grid points obtained by using (11).*

Then, the estimate:

$|y(x_j) - v_j| \le Mh^2$, $j=0(1)n+1$, where M is a constant independent of ε and h, holds.

Proof. We shall estimate separately the truncation error for each function u, w and g (see [4]).

Since, $\tau_i(u) = 0$, $\tau_i(w) = 0$ for $q(x) = q(0) = \text{const}$, we have

$$\tau_i(u) = \tau_i(u) - \bar{\tau}_i(u), \; \tau_i(w) = \tau_i(w) - \bar{\tau}_i(w),$$

where $\bar{\tau}_i(u)$, $\bar{\tau}_i(w)$ are the expressions $\tau_i(u)$ and $\tau_i(w)$ with $q(x) = q(0) = \text{const}$.

After some Taylor developments in the corresponding expressions from Lemma 1, we obtain that

$$|\tau_i(u)|, \; |\tau_i(w)| \le Mh^3/\varepsilon \quad \text{when} \quad h^2 \le \varepsilon.$$

For $\tau_i(g)$ we also use the form given by Lemma 1, and from Lemma 3 we have that

$$|\tau_i(g)| \le Mh^3/\varepsilon, \quad h^2 \le \varepsilon.$$

Thus, (Lemma 3) we have

$$|\tau_i(y)| \le Mh^3/\varepsilon \quad \text{when} \quad h^2 \le \varepsilon. \tag{12}$$

Let $\varepsilon \le h^2$. Then we use the truncation errors for u, w and g in the form given in Lemma 2.

Since all the coefficients in scheme (11) are bounded by Mh^{-1}, we obtain directly that

$$|\tau_i(g)| \le Mh.$$

Using the fact that

$$\tau_1(u) = \tau_1(u) - \bar{\tau}(u), \quad \tau_1(w) = \tau_1(w) - \bar{\tau}(u),$$

$|q(x_1) - q(0)| \le Mx_1^2$, from Lemma 2 and Lemma 3 we have that

$$|\tau_1(u)|,\ |\tau_1(w)| \le Mh, \text{ when } \varepsilon \le h^2.$$

Thus,

$$|\tau_1(y)| \le Mh, \quad \text{for } \varepsilon \le h^2. \tag{13}$$

Denote by A the matrix of system (11). Since,

$$|y(x_1) - v_1| \le \|A^{-1}\| \max_1 |\tau_1(y)| \text{ and}$$

$$\|A^{-1}\| \le \max_1 |r^c + r^- + r^+|^{-1} \le \begin{cases} M\varepsilon/h, & h^2 \le \varepsilon, \\ Mh, & \varepsilon \le h^2, \end{cases}$$

from (12) and (13) we have that Theorem 1 is valid.

5. Numerical results

Our test for the order of uniform convergence, notation and example are taken from [1]

Example:

$$-\varepsilon y'' + y = -\cos\pi x - 2\varepsilon\pi^2\cos 2\pi x,$$

$$y(0) = 0, \quad y(1) = 1$$

Exact solution:

$$y(x) = \exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon})/(1+\exp(-1/\sqrt{\varepsilon})) - \cos^2\pi x$$

Table 1 contains the estimate for the order of uniform convergence.

Table 1

ε \ k	0	1	2	3	4	$\overline{P}\varepsilon$
1/2	2.02	2.00	2.00	2.00	2.00	2.00
1/4	2.02	2.00	2.00	2.00	2.00	2.00
1/8	2.02	2.01	2.00	2.00	2.00	2.01
1/16	2.03	2.01	2.00	2.00	2.00	2.01
1/32	2.03	2.01	2.00	2.00	2.00	2.01
1/64	2.05	2.01	2.00	2.00	2.00	2.01
1/128	2.08	2.02	2.01	2.00	2.00	2.02
1/256	2.14	2.04	2.01	2.00	2.00	2.04
1/512	2.23	2.07	2.02	2.01	2.00	2.06

The computed order of uniform convergence is 2.06 and the classical one is 2.00.

Table 2 contains the maximal differences between the exact and approximate solution at the points of the grid.

Table 3 contains the same results for the cubic spline difference scheme, which is given in [4].

Table 2

$$\max_{i} |y(x_i) - v_i|$$

ε \ n	16	32	64	128
1/2	.58E-2	.14E-2	.36E-3	.90E-4
1/4	.54E-2	.13E-2	.33E-3	.82E-4
1/8	.48E-2	.12E-2	.29E-3	.73E-4
1/16	.41E-2	.10E-2	.25E-3	.63E-4
1/32	.37E-2	.90E-3	.22E-3	.56E-4
1/64	.34E-2	.84E-3	.21E-3	.52E-4
1/128	.34E-2	.82E-3	.20E-3	.50E-4
1/256	.36E-2	.83E-3	.20E-3	.50E-4
1/512	.38E-2	.84E-3	.20E-3	.50E-4

256	512	1024
.22E-4	.56E-5	.14E-5
.20E-4	.51E-5	.13E-5
.18E-4	.45E-5	.11E-5
.16E-4	.39E-5	.91E-6
.14E-4	.35E-5	.87E-6
.13E-4	.32E-5	.81E-6
.13E-4	.32E-5	.79E-6
.13E-4	.31E-5	.78E-6
.13E-4	.31E-5	.78E-6

Table 3.

ε \ n	32	64	128	256	512
1/2	0.29E-2	0.72E-3	0.18E-3	0.45E-4	0.11E-4
1/4	0.26E-2	0.66E-3	0.17E-3	0.41E-4	0.10E-4
1/8	0.23E-2	0.59E-3	0.15E-3	0.37E-4	0.92E-5
1/16	0.20E-2	0.51E-3	0.13E-3	0.32E-4	0.79E-5
1/32	0.18E-2	0.45E-3	0.11E-3	0.28E-4	0.70E-5
1/64	0.18E-2	0.42E-3	0.10E-3	0.26E-4	0.65E-5
1/128	0.16E-2	0.40E-3	0.10E-3	0.20E-4	0.53E-5
1/256	0.16E-2	0.40E-3	0.10E-3	0.25E-4	0.63E-5
1/512	0.16E-2	0.40E-3	0.10E-3	0.25E-4	0.63E-5

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Rezime

SPLAJN DIFERENCNA ŠEMA SA KVADRATNIM SPLAJNOVIMA
ZA SAMO-ADJUNGOVANI KONTURNI PROBLEM

Izvedena je eksponencijalno fitovana diferencna šema zasnovana na kvadratnim splajnovima za problem: $-\varepsilon y'' + \alpha(x)y = f(x)$, $0 < x < 1$, $y(0) = \alpha_0$, $y(1) = 1$. Šema ima drugi red tačnosti pod odredenim uslovima na funkcije q i f. Dati su numerički rezultati.

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