Univ. u Novom Sadu
Zb.Rad. Prirod.-Mat.Fak.
Ser.Mat. 19,2, 149-160 (1989)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

SPLINE DIFFERENCE SCHEME ON A NON-UNIFORM MESH

Surla K.*, Jerkovic V.**

* Institute of Mathematics, Dr Ilije Duričića 4, 21000 Novi Sad, Yugoslavia
** Radio-Television Novi Sad, Žarka Zrenjanina 3, 21000 Novi Sad, Yugoslavia

Abstract

The uniformly convergent cubic spline difference scheme for the problem $Ly = -\varepsilon y'' + q(x)y = f(x)$, $0 < x < 1$, $q(x) \ge q > 0$, $y(0) = \alpha_0$, $y(1) = \alpha_1$ on a non-uniform mesh is derived. The scheme provides for the location of a larger number of points in the boundary layers, while the order of accuracy and the structure of the matrix remain the same as in the uniform grid.

1. Introduction

In [3] a spline difference scheme was constructed on a uniform grid for the problem:

$$
(1)\qquad \begin{aligned} Ly &= -\varepsilon y'' + q(x)y = f(x), \quad 0 < x < 1, \\ y(0) &= \alpha_0, \quad y(1) = \alpha_1, \quad q(x) \ge q > 0. \end{aligned}
$$

It has been proved that the scheme has a second order of uniform accuracy under the condition $q'(0) = q'(1) = 0$. Schemes of the same order of accuracy on a uniform mesh are given in [1], [2] for this problem. Since the spline difference schemes have the same order of precision and the same matrix structure on the uniform and on the non-uniform grid for a fixed ε, we wanted to use this characteristic for singularly perturbed problems. This would enable the modifying of the distribution of the point to the characteristics of the exact solution. Since the condition for the inverse

AMS Mathematics Subject Classification (1980): 65L10.
Key words and phrases: Spline difference scheme, fitting factor, singular perturbation problem.

monotone of the matrix in ordinary spline difference schemes ([6]) is $h_i^2 q_j/6c$, $i=j-1,j$ in [3] the exponential fitting factor was introduced along with the second derivation and the condition was eliminated for the equidistant grid. The determination of the fitting factor for the non-equidistant grid is more complicated. In [3] the fitting factor is determined in such a way that the trunction error of the difference scheme for the function of the boundary layer should be zero. The determining of special conditions for the grid provides for the second order of accuracy.

Proof will be given here of the uniform convergence sheme [3] along with the numerical results which point out the advantage of the non-equidistant grid. The case $q'(0) \neq q'(1)$ will be considered.

2. Derivation of the scheme

Let us substitute differential equations (1) by the equation:

$$- \sigma(x)y'' + q(x)y = f(x), \tag{2}$$

where $\sigma(x)$ is the fitting factor which will be determined subsequently. The approximate solution of equation (2) should be sought in the form of a cubic spline $v(x) \in C^2\,[0,1]$:

$$v(x) = v_j + (x-x_j)v_j^{(1)} + \frac{(x-x_j)^2}{2} v_j^{(2)} + \frac{(x-x_j)^3}{6} v_j^{(3)},$$

$$x \in [x_j, x_{j+1}], \quad 0 = x_0 < x_1 < x_2 \dots x_n < x_{n+1} = 1.$$

The points of collocation are the points of the spline grid. Constants are determined from the system of equations:

$$- \sigma_j v_j^{(2)} + q_j v_j = f_j, \quad j = 0(1)n+1, \tag{3}$$

$$v_j = v_{j-1} + hv_{j-1}^{(1)} + \frac{h^2}{2} v_{j-1}^{(2)} + \frac{h^3}{6} v_{j-1}^{(3)}, \quad j = 1(1)n, \tag{4}$$

$$v_j^{(1)} = v_{j-1}^{(1)} + hv_{j-1}^{(2)} + \frac{h^2}{2} v_{j-1}^{(3)}, \quad j = 1(1)n, \tag{5}$$

$$v_j^{(2)} = v_{j-1}^{(2)} + hv_{j-1}^{(3)}, \quad j = 1(1)n, \tag{6}$$

$$v(0) = \alpha_0, \quad v(1) = \alpha_1.$$

Equation (3) expresses the weakened condition of collocation. Equations (4), (5) and (6) are obtained from the continuity of function $v(x)$. By the elimination of $v_j^{(1)}$, $v_j^{(2)}$, and $v_j^{(3)}$ from the above equations (see [4]) we get the difference scheme:

(7)
$$R_h v_j = Q_h f_j, \quad j = 1(1)n,$$

$$R_h v_j = - r_j^- v_{j-1} + r_j^c v_j - r_j^+ v_{j+1},$$

$$Q_h f_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1},$$

$$r_j^- = \left(1 - \frac{h_{j-1} q_{j-1}}{6\sigma_{j-1}}\right) \frac{1}{h_{j-1}}, \quad r_j^+ = \left(1 - \frac{h_j^2 q_{j+1}}{6\sigma_{j+1}}\right) \frac{1}{h_j},$$

$$r_j^c = \left(1 + \frac{h_j^2 q_j}{3\sigma_j}\right) \frac{1}{h_j} + \left(1 + \frac{h_{j-1}^2 q_j}{3\sigma_j}\right) \frac{1}{h_{j-1}},$$

$$q_j^- = \frac{h_{j-1}}{6\sigma_{j-1}}, \quad q_j^c = \frac{h_{j-1} + h_j}{3\sigma_j}, \quad q_j^+ = \frac{h_j}{6\sigma_{j+1}},$$

$$v_0 = \alpha_0, \quad v_{n+1} = \alpha_1, \quad q_j = q(x_j), \quad f_j = f(x_j).$$

In order to get a suitable fitting factor σ_j, we shall use the following lemma.

Lemma 1 [1] *Let* $y \in C^4[0,1]$. *Let* $q'(0) = q'(1) = 0$. *Then the solution of problem* (1) *has the form:*

$y(x) = u(x) + w(x) + g(x)$, *where*

$u(x) = p_0 \; exp(-x\sqrt{q(0)/\varepsilon})$,

$w(x) = p_1 \; exp(-(1-x) \sqrt{q(1)/\varepsilon})$,

p_0 *and* p_1 *are bounded functions of* ε *independent of* x *and:*

$$|g^{(j)}| \le M\,(1 + \varepsilon^{1-(1/2)j}), \quad j = 0(1)4.$$

M *is a constant independent of* ε.

The local truncation error of scheme (7) for an arbitrary smooth function φ has the form:

$$\tau_j(\varphi) = R_h \varphi_j - Q_h (L\varphi)_j \; .$$

Lemma 2. *Let in scheme (7)* σ_{j-1} *be replaced by* σ_j^-, σ_j *by* σ_j^c *and* σ_{j+1} *by* σ_j^+ *where:*

$$\sigma_j^- = \frac{h_{j-1}^2 q_{j-1}}{6} S_j(t),$$

$$\sigma_j^c = \frac{h_j h_{j-1}}{6} q_j S_j(t),$$

$$\sigma_j^+ = \frac{h_j^2 q_{j+1}}{6} S_j(t),$$

$$S_j(t) = 1 + \frac{3(h_j + h_{j-1})}{s_j(t)},$$

$s_j(t) = h_j \exp(h_{j-1}t) - (h_j + h_{j-1}) + h_{j-1} \exp(-h_j t)$, *for* $j = 0(1)i_1$,

$s_j(t) = h_j \exp(-h_{j-1}t) - (h_j + h_{j-1}t) + h_{j-1} (\exp(h_j t)$ *for* $i_2 \le j \le n+1$,

$t = \sqrt{q_j/\varepsilon}$, $i_1 \le i_2$, *then:*

a) $\tau_j(u) = 0$, $j = 0(1)i_1$, *for* $q(x) = q = const.$

b) $\tau_j(w) = 0$, $j = i_1(1)n+1$, $i_1 \le i_2$ *for* $q(x) = q = const.$

c) *The matrix of system* (7) *is an inverse monotone.*

Proof. a), b) By the direct substitution of σ_j, σ_{j-1} and σ_{j+1} in the expression $\frac{h_j h_{j-1}}{h_j + h_{j-1}} R_h u_j$ we obtain zero. Since $Lu_j = 0$, it follows that $\tau_j(u) = 0$. In the same manner it can proved that $\tau_j(w) = 0$.

c) Since $r^- \ge 0$, $r^+ \ge 0$, $r^c > r^- + r^+$ the proof follows.

Throughout the paper M and δ will denote different constants independent of ε and h_j.

Lemma 3. *Let:*

$$(9) \quad \begin{cases} 0 \le h_j - h_{j-1} \le M_j \dfrac{h_{j-1}}{\varepsilon} \min(M_0 h_{j-1}^2, \varepsilon), & 1 \le j \le i_1, \\ 0 \le h_{j-1} - h_j \le M_j \dfrac{h_j}{\varepsilon} \min(M_0 h_j^2, \varepsilon), & i_2 \le j \le n, \end{cases}$$

$$\frac{h_j}{h_{j\pm1}} \le M, \quad h_j = h = const. \qquad i_1 \le j < i_2\ ,\ M_i \le M,\ i=0(1)n.$$

Then, the estimate:

$$\max\left(|\sigma_j^- -\varepsilon|,\ |\sigma_j^c -\varepsilon|, |\sigma_j^+ -\varepsilon|\right) \le Mh_j^2 \text{ holds.}$$

Proof. Let $\varepsilon \le M_0 h_j^2$. Then:

$$\left|\frac{h_j+h_{j-1}}{s_j(t)}\right| \le M \exp(-h_{j-1}t),$$

$$|S_j(t)| \le M, \quad \frac{h_j}{h_{j\pm1}} \le M, \quad |\sigma_j^c| \le Mh_j^2,$$

$$|\sigma_j^c-\varepsilon| \le Mh_j^2.$$

Let $M_0 h_j^2 \le \varepsilon$. Then, after some Taylor developments we obtain:

$$\frac{3(h_j+h_{j-1})}{s(q_j)} = \frac{6}{h_j h_{j-1} q_j} + 2\,\frac{h_j-h_{j-1}}{h_j h_{j-1}}\sqrt{\frac{\varepsilon}{q_i}} + O(1),$$

$$|\sigma_j^c-\varepsilon| \le Mh_j^2 + M|h_j-h_{j-1}|\sqrt{\varepsilon} \le Mh_j^2 .$$

Since $\frac{h_j}{h_{j-1}} = 1 + \frac{M}{\varepsilon}\min(M_0 h_{j-1}^2, \varepsilon)$ we have $|\sigma_j^- -\varepsilon| \le Mh_j^2$ and $|\sigma_j^+ -\varepsilon| \le Mh_j^2$.

3. Proof of the uniform convergence

Lemma 4. *Let $y \in C^4[0,1]$. Then, the truncation error of scheme (7) has the form:*

$$\tau_j(y) = R_h(y_j - v_j) = -\frac{\varphi_{2,j+1}}{h_j} + \frac{\varphi_{2,j}}{h_{j-1}} - \varphi_{1,j}, \tag{10}$$

$$\varphi_{1,j} = \psi_{1,j} - \frac{h_{j-1}\psi_{2,j}}{2} + \frac{h_{j-1}}{2}\left(\frac{\eta_{j-1}}{\sigma_{j-1}} + \frac{\eta_j}{\sigma_j}\right),$$

$$\varphi_{2,j} = \psi_{0,j} + h_{j-1}^2\left(\frac{\eta_{j-1}}{3\sigma_{j-1}} + \frac{\eta_j}{6\sigma_j} - \frac{\psi_{2,j}}{6}\right),$$

$$\eta_j = y_j''(\sigma_j - \varepsilon),$$

$$\psi_{k,j} = \frac{h_{j-1}^{4-k}}{(4-k)!} y^{IV}(\xi_{k,j}), \quad x_{j-1} \le \xi_{k,j} \le x_j.$$

Proof. See [4].

Lemma 5. *Let* $q'(0) = q'(1) = 0$ *in* (1). *Let* $y(x) \in C^4[0,1]$. *Let in (7)* σ_j, σ_{j-1} σ_{j+1} *be replaced by Lemma 2.*

Then:

$$|\tau_j(y)| \le M \frac{h_j}{\varepsilon} \min(M_0 h_j^2, \varepsilon). \tag{11}$$

Proof. According to Lemma 1 we have:

$$\tau_j(y) = \tau_j(u) + \tau_j(w) + \tau_j(g). \tag{12}$$

We shall estimate separately the truncation error of each function u, w, g. We shall start with $u=u(x)$.

Denote by $\tilde{\tau}_j(u)$ the truncation error $\tau_j(u)$ in the case $q(x) = q_0 = q(0) =$ const. The corresponding expressions in $\tilde{\tau}_j(u)$ we mark by ~. Since $\tilde{\tau}_j(u) = 0$ (Lemma 2), we have:

$$\tau_j(u) = \tau_j(u) - \tilde{\tau}_j(u) = \frac{1}{h_j}(-\varphi_{2,j+1} + \tilde{\varphi}_{2,j+1}) + \frac{1}{h_{j-1}}(\varphi_{2,j} - \tilde{\varphi}_{2,j}) -$$
$$- \varphi_{1,j} + \tilde{\varphi}_{1,j},$$

where every σ_{j-1}, σ_j and σ_{j+1} are replaced by σ_j^-, σ_j^c and σ_j^+ respectively.

Let $M_0 h_j^2 < \varepsilon$, then:

$$\left|\frac{1}{\tilde{\sigma}_j^c} - \frac{1}{\sigma_j^c}\right| \le \frac{1}{\tilde{\sigma}_j^c \sigma_j^c}\left[\frac{h_j h_{j-1}}{6}(q_j - q_0) + \frac{h_j - h_{j-1}}{12}(\sqrt{q_j} - \sqrt{q_0})\sqrt{\varepsilon} +\right.$$
$$\left. + O(h_j^2)(q_j - q_0)\right] \le M\varepsilon^{-2} h_j^2 x_j^2,$$

$$|\varphi_{j,1} - \tilde{\varphi}_{j,1}| \le M \frac{h_{j-1}}{\varepsilon}\frac{h_j^2}{\varepsilon^2} x_j^2 \exp(-x_j\sqrt{q_0/\varepsilon}) \le M \frac{h_j}{\varepsilon} h_j^2 \exp(-\delta x_j \sqrt{\varepsilon}),$$

$$|\varphi_{j,2} - \tilde{\varphi}_{j,2}| \le h^2_{j-1} \left| \frac{\eta^-_j}{3\sigma_{j-1}} - \frac{\tilde{\eta}^-_j}{3\sigma_{j-1}} + \frac{\eta^c_j}{6\sigma_j} - \frac{\tilde{\eta}^c_j}{6\sigma_j} \right| \le Mh^2_{j-1} \frac{h^2_j}{\varepsilon} ,$$

$|\tau_j(u)| \le M\,(h^2_j/\varepsilon)$, where

$$\eta^-_j = y''_{j-1}(\sigma^-_j - \varepsilon) \quad , \quad \eta^c_j = y''_j(\sigma^c_j - \varepsilon) \ , \quad \eta^+_j = y''_{j+1}(\sigma^+_j - \varepsilon).$$

In the same way we can conclude that:

$$|\tau_j(w)| \le M \frac{h^3_{j-1}}{\varepsilon} . \tag{13}$$

For $\tau_j(g)$ we have $\eta^{\pm c}_j = g''_j(\sigma^{\pm c}_j - \varepsilon) = O(h^2_j)$ and from (10) and Lemma 1 we can see that:

$|\tau_j(g)| \le M \frac{h^3_j}{\varepsilon}$, for $M_0 h^2_j \le \varepsilon$, and (11) holds.

Let $\varepsilon \le M_0 h^2_j$. Then:

$$|s_j(t)| \le Mh_j \exp(h_j t),$$

$$\left|\frac{1}{s_j(t)} - \frac{1}{\tilde{s}_j(t)}\right| \le \frac{M}{h^2_j \exp\left[\frac{h_j}{\sqrt{\varepsilon}}(q_0 - q_j)\right]} \left(\frac{h^2_{j-1} x^2_j}{\sqrt{\varepsilon}} \exp(-\eta_1) + \frac{h^2_j x^2_j}{\sqrt{\varepsilon}} \exp(\eta_2)\right),$$

where η_1 is a point between $h_{j-1}\sqrt{q_j/\varepsilon}$, $h_{j-1}\sqrt{q_0/\varepsilon}$, η_2 is a point between $h_j\sqrt{q_j/\varepsilon}$ and $h_j\sqrt{q_0/\varepsilon}$.

Furthermore:

$$\left|\frac{1}{\sigma^c_j} - \frac{1}{\tilde{\sigma}^c_j}\right| \le M\, h^{-2}_j x^2_j + \frac{x^2_j}{\sqrt{\varepsilon}} ,$$

$$|\varphi_{j,1} - \tilde{\varphi}_{j,1}| \le \varepsilon \ \frac{h_j}{2} u''_j \left|\frac{1}{\sigma^c_j} - \frac{1}{\tilde{\sigma}^c_j}\right| + \varepsilon\, \frac{h_{j-1}}{2} u''_{j-1} \left|\frac{1}{\sigma^-_{j-1}} - \frac{1}{\tilde{\sigma}^-_{j-1}}\right|,$$

$$|\varphi_{j,1} - \tilde{\varphi}_{j,1}| \le M\sqrt{\varepsilon}, \quad |\varphi_{j,2} - \tilde{\varphi}_{j,2}| \le M\, h_j\sqrt{\varepsilon}.$$

According to Lemma 4, by the given estimates we have:

$$|\tau_j(u)| \le M\sqrt{\varepsilon} . \tag{14}$$

In a similar manner we obtain:

$$|\tau_j(w)| \le M\sqrt{\varepsilon} . \tag{15}$$

For estimating the truncation error corresponding to function g we use the form:

$$T_j(g) = T_0 g_j + T_1 g'_j - r_j^- \frac{h_{j-1}^2}{2} g''(b_1) - r_j^c + r_j^+ \frac{h_j^2}{2} g''(b_2) -$$

$$- q_j^- q_{j-1} \frac{h_{j-1}^2}{2} g''(b_1) - q_j^+ \frac{h_j^2}{2} g''(b_2) q_{j+1} +$$

$$+ \varepsilon (q_j^- g''_{j-1} + q_j^c g''_j + q_j^+ g''_{j+1}), \quad \text{where} \quad x_{j-1} \le b_1 \le x_j \le b_2 \le x_{j+1} .$$

Since $T_0=0$, $T_1=0$, $|r^{\pm c}| \le M h_j^{-1}$, $|q^{\pm c}| \le M h_j^{-1}$, we have that:

$$|\tau_j(g)| \le Mh_j . \tag{16}$$

Then from (14), (15), (16) and Lemma 1 we obtain (11).

Theorem 1. *Let conditions of Lemma 1 be fulfilled. Let v_j, $j= 0(1)n+1$ be the approximation to the solution $y(x)$ of (1) obtained by using (7) with σ_j, σ_{j-1}, σ_{j+1} determined by Lemma 2. Let points x_j, $j=0(1)n+1$ satisfy conditions (9).*

Then the estimate:

$$|y(x_j) - v_j| \le Mh_j^2, \quad j = 0(1)n+1 \quad \text{holds}. \tag{17}$$

Proof. Let A be the matrix of system (7). A is an inverse monotone matrix and the discrete maximum principle holds. Since:

$$- r_j^- + r_j^c - r_j^+ \ge \begin{cases} Mh_j/\varepsilon, & M_0 h_j \ge \sqrt{\varepsilon}, \\ Mh_j^{-1}, & M_0 h_j \le \sqrt{\varepsilon}, \end{cases}$$

from (11) it follows that:

$R_h(\pm (y(x_j) - v_j) + Mh^{-2}) \ge 0, h^- = max(h_j, h_{j-1}) = const.$, and (17) holds.

In the case $q'(0) \ne q'(1)$, we can prove only the first order of the uniform convergence.

Theorem 2. *Let $y(x) \in C^4[0,1]$. Let v_j, $j=0(1)n+1$ be the approximation to the solution $y(x)$ of (1) obtained by using (7) with σ_{j-1}, σ_j, σ_{j+1} determined by Lemma 2.*

Let points x_j, $j=0(1)n+1$, satisfy conditions (9).

Then:

$$|y(x_j) - v_j| \le Mh_j, \quad j = 0(1)n+1.$$

Proof. Function $g(x)$ from Lemma 1 (see [5]) satisfies the following inequality:

$$|g^{(j)}(x)| \le M\,(1 + \varepsilon^{1/2(1-j)}), \quad j = 0(1)4.$$

If we repeat the entire proof of Theorem 1 for this estimate of derivations for $g(x)$ and if we use:
$|q(x_j) - q(0)| \le Mx_j$, we shall obtain the statement of Theorem 2.

4. Numerical results

Scheme (7) was used to obtain the approximate solution of problem [1]:

$$-\varepsilon y'' + (1 + x(1-x))y = f(x)$$

$$y(0) = y(1) = 0.$$

Its exact solution is:

$$y(x) = 1 - (1-x)\,\exp(-x/\sqrt{\varepsilon}) - \exp((x-1)/\sqrt{\varepsilon}).$$

Here we shall present the numerical results which suggest that scheme (7) achieves a uniform second-order accuracy. These results also suggest the choice of the grid.

Table 1, 2 and 3 contain the maximal differeces between the exact and approximate solution at the points of the grid, for different ε and n.

Table 1 contains the results on the equidistant grid.

Table 1

$$\max_j |v_j - y(x_j)|$$

ε \ $n+1$	32	64	128	256	512
1/2	0.34-04	0.84-05	0.21-05	0.55-06	0.16-06
1/4	0.58-04	0.14-04	0.36-05	0.93-06	0.25-06
1/8	0.80-04	0.20-04	0.50-05	0.13-05	0.34-06
1/16	0.86-04	0.21-04	0.54-05	0.14-05	0.36-06
1/32	0.99-04	0.25-04	0.62-05	0.16-05	0.41-06
1/64	0.13-03	0.33-04	0.82-05	0.21-05	0.54-06
1/128	0.19-03	0.46-04	0.12-04	0.29-05	0.75-06
1/256	0.26-03	0.66-04	0.16-04	0.41-05	0.10-05
1/512	0.42-03	0.95-04	0.23-04	0.58-05	0.15-05
1/1024	0.62-03	0.13-03	0.33-04	0.81-05	0.20-05
1/2048	0.88-03	0.21-03	0.47-04	0.11-04	0.29-05
1/16384	0.18-02	0.60-03	0.15-03	0.33-04	0.83-05

The non-equidestant grid is formed in such a way that more points are found in the layers than outside them.

On the interval $[0,c_0]$, c_0 is a constant which is given in advance, the grid is non-equidistant and obtained according the formula:

$$\tilde{h}_j = \tilde{h}_{j-1} + M\,\frac{\tilde{h}_{j-1}}{\varepsilon}\,\min(\tilde{h}^2_{j-1},\varepsilon) \quad j = 1(1)n_1-1,$$

$$h_j = Q\tilde{h}_j,\ Q = c_0/\tilde{Q},\quad \tilde{Q} = \sum_{j=1}^{n_1-1}\tilde{h}_j,\quad x_j = x_{j-1} + h_{j-1},\ j = 1(1)n_1,\quad x_0 = 0,$$

$\tilde{h}_0$, M and c_0 are given.

On the interval $[c_0,1-c_0]$ the grid is equidistant:

$$h_j = (1-2c_0)/n_2,\quad j = n_1(1)n_1+n_2-1,$$

$$n+1 = 2n_1+n_2,\quad x_j = x_{j-1}+h_j,\quad j = n_1+1(1)n_1+n_2-1.$$

On the interval $[1-c_0,1]$ the grid is symmetric to the grid on interval $[0,c_0]$. This is a starting mesh. In each next step the invervals are halved.

Table 2.

$c_0 = 0.05$, $\tilde{h}_0 = 0.03$, $\tilde{M} = 110.5$, $n_1/(n+1) = 1/8$

$$\max_j |v_j - y(x_j)|$$

ε \ n+1	32	64	128	256	512
1/2	0.39-04	0.99-05	0.25-05	0.65-06	0.18-06
1/4	0.82-04	0.21-04	0.52-05	0.13-05	0.35-06
1/8	0.13-03	0.32-04	0.79-05	0.20-05	0.52-06
1/16	0.16-03	0.39-04	0.97-05	0.25-05	0.64-06
1/32	0.21-03	0.54-04	0.13-05	0.34-05	0.87-06
1/64	0.29-03	0.75-04	0.19-05	0.48-05	0.12-05
1/128	0.39-03	0.99-04	0.25-04	0.64-05	0.16-05
1/256	0.46-03	0.11-03	0.29-04	0.74-05	0.19-05
1/512	0.47-03	0.11-03	0.28-04	0.70-05	0.12-05
1/1024	0.45-03	0.98-04	0.24-04	0.60-05	0.15-05
1/2048	0.57-03	0.13-03	0.31-04	0.77-05	0.19-05
1/16384	0.12-02	0.37-03	0.10-03	0.24-04	0.62-05

(About 25 % points in the layers)

Tabla 3.

$c_0 = 0.05$, $\tilde{h}_0 = 0.01$, $\tilde{M} = 12$, $n_1/(n+1) = 1/4$

$$\max_j |v_j - y(x_j)|$$

ε \ $n+1$	32	64	128	256	512
1/2	0.84-04	0.22-04	0.54-04	0.14-05	0.38-06
1/4	0.18-03	0.46-04	0.12-04	0.29-05	0.76-06
1/8	0.28-03	0.71-04	0.18-04	0.45-05	0.11-05
1/16	0.34-03	0.88-04	0.22-04	0.57-05	0.14-05
1/32	0.47-03	0.13-03	0.32-04	0.83-05	0.21-05
1/64	0.63-03	0.18-03	0.47-04	0.12-04	0.31-05
1/128	0.77-03	0.22-03	0.61-04	0.16-04	0.41-05
1/256	0.84-03	0.24-03	0.66-04	0.17-04	0.45-05
1/512	0.86-03	0.22-03	0.58-04	0.15-04	0.39-05
1/1024	0.62-03	0.13-03	0.32-04	0.82-05	0.21-05
1/2048	0.51-03	0.14-03	0.37-04	0.97-05	0.25-05
1/16384	0.63-02	0.26-03	0.88-04	0.27-04	0.75-05

(About 50 % points in the layers)

From this example it can be seen that about 50 % of the points can be located in the boundary layers while the error remains approximately the same as on the equidistat grid. In addition to this, it can be seen that the results on the nonequidistant grid are better for smaller ε.

References

1. **Doolan, E.P., Miller, J.J., Schilders, W.H.A.:** Uniform numerical methods for problems with initial and boundary layers. Dublin:Boole Press 1980.
2. **Hegarty, A.F., Miller, J.J.H., O'Riordan, E.:** Uniform second order difference schemes for singular perturbation problems (J.J.H. Miller, ed.), Boundary and Interior Layers - Computational and Asumptotic Methods, Boole Press, Dublin, 1980, 301-305.
3. **Surla, K.:** A Uniformly Convergent Spline Difference Scheme for a Singularly Perturbed Self-adjoint Problem on a Non-uniform Mesh, Z. angew. Math. Mech. 66 (1986) T328-T330.
4. **Surla, K., Stojanovic, M.:** Singularly Perturbed Spline Difference Schemes on a Non-equidistant Grid, Z. angew. Math. Mech. 68 (1988)3, 171-180.

5. **Vulanovic, R.**: Mesh Contruction for the Discretization of Singularly Perturbed Boundary Value Problems, Doctoral Dissertation, Faculty of Science, University of Novi Sad, 1986.

6. **Zavjalov, Ju.S., Kvasov, B.I., Mirosnicenko, Z.L.**: Metody splajn funkcii, Nauka, Moskva, 1980.

Rezime

SPLAJN DIFERENCNA ŠEMA NA NERAVNOMERNOJ MREŽI

Izvedena je uniformno konvergentna diferencna šema zasnovana na kubnim splajnovima na ravnomernoj mreži za problem $-\varepsilon y''+q(x)y=f(x)$, $0<x<1$, $q(x)\geq q>0$, $y(0)=\alpha_0$, $y(1)=\alpha_1$. Šema omogućuje smeštanje većeg broja tačaka u granične slojeve, pri čemu red tačnosti i struktura matrice ostaju isti kao na ravnomernoj mreži.

Received by the editours November 6, 1987.