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CONVERGENCE OF A SEQUENCE OF GENERALIZED RANDOM PROCESSES ON THE ZEMANIAN SPACE $\mathcal{A}$

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Abstract

Different types of convergences of a sequence of generalized random processes on the Zemanian space $\mathcal{A}$ are defined and compared.

1. Introduction

Generalized random processes (g.r.p.) were defined by several authors [1,2,4,5,7,8,9,10,11,12]. Different types of convergences of a sequence of g.r.p.-s were introduced and investigated in [4]. In [1,2,4,5,7,10,11,12] spaces $\mathcal{D}$ and $K\{M_p\}$ were taken to be the spaces of test functions and in [8,9] the Zemanian space $\mathcal{A}$. In [1,4,5,7,8,9,10,12] the representation theorems for a g.r.p. were obtained.

For a space of test functions we take the space $\mathcal{A}$, whose elements have an orthogonal expansion. The space $\mathcal{A}$ and its dual space $\mathcal{A}'$ were introduced in [13]. Our construction of the spaces $\mathcal{A}$ and $\mathcal{A}'$ is different from [13], and the details are given in [8].

In [4] the representation theorems for a sequence of g.r.p. on $K\{M_p\}$ converging almost surely, in probability and mean (K') were obtained. Following [4], in [9] representation theorems for a sequence of g.r.p.-s on converging almost surely $(\mathcal{A}')$ are obtained.

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In Sections 2., 3. we shall give the basic definitions and properties of space $\mathcal{A}$ and of a g.r.p. on $\mathcal{A}$. In Section 4. we shall define various types of convergences of a sequence of g.r.p.-s and give representation theorems for sequence of g.r.p.-s converging in probability and mean ($\mathcal{A}'$).

2. Spaces $\mathcal{A}$ and $\mathcal{A}'$

We shall use the notation from [13]. Let I be an open interval of the real line $\mathbb{R}$ and $L^2(I)$ be the spaces of the equivalence classes of square integrable functions with values in the set of complex numbers, $\mathbb{C}$, with the usual norm. Denote by $C^\infty(I)$ the set of infinitely differentiable (smooth) functions, by $\mathbb{N}$ the set $\{1,2,\dots\}$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{R}$ be a linear differential self-adjoint operator of the form

$$\mathcal{R} = \theta_0 D^{n_1} \theta_1 \dots D^{n_\nu} \theta_\nu,$$

such that

$$\mathcal{R} = \bar{\theta}_\nu (-D)^{n_\nu} \dots (-D)^{n_2} \bar{\theta}_1 (-D)^{n_1} \bar{\theta}_0 ,$$

where $D=d/dx$, $n_k \in \mathbb{N}_0$, $k=1,2,\dots,\nu$, θ_k, $k=0,1,\dots,\nu$, are smooth functions without zeros on I, and $\bar{\theta}_k$ are complex conjugates of θ_k, $k=0,1,\dots,\nu$. We suppose that there exist a sequence of real numbers $\{\lambda_n,\ n \in \mathbb{N}_0\}$, and a sequence of smooth functions $\{\psi_n,\ n \in \mathbb{N}_0\}$ such that $\mathcal{R}\psi_n = \lambda_n \psi_n$, $n \in \mathbb{N}_0$. Furthermore, suppose that the sequence $\{\lambda_n,\ n \in \mathbb{N}_0\}$ monotonically tends to infinity and that $\{\psi_n,\ n \in \mathbb{N}_0\}$ forms a complete orthonormal system in $L^2(I)$. We can enumerate the sequences $\{\lambda_n,\ n \in \mathbb{N}_0\}$ and $\{\psi_n,\ n \in \mathbb{N}_0\}$, so that $|\lambda_0| \le |\lambda_1| \le |\lambda_2| \le \dots$. Put $\tilde{\lambda}_n = \lambda_n$ if $\lambda_n \neq 0$ and $\tilde{\lambda}_n = 1$, if $\lambda_n = 0$, $n \in \mathbb{N}_0$. The sequence $\{\tilde{\lambda}_n,\ n \in \mathbb{N}_0\}$ is nondecreasing and $|\tilde{\lambda}_n| \to \infty$. Let $\mathcal{R}^{k+1} = \mathcal{R}(\mathcal{R}^k)$, $k \in \mathbb{N}_0$, where $\mathcal{R}^0 = \mathcal{J}$, $\mathcal{J}$ is the identity operator. In [8], the scale of spaces $\mathcal{A}_k$, $k \in \mathbb{N}_0$ is defined in the following way:

$$\mathcal{A}_k = \left\{ \phi \in L^2(I):\ \phi = \sum_{m=0}^{\infty} a_m \psi_m,\ \|\phi\|_k = \sum_{m=0}^{\infty} |a_m|^2 |\tilde{\lambda}_m|^{2k} < \infty \right\}, \quad k \in \mathbb{N}_0$$

Put

$$\mathcal{A} = \bigcap_{k=0}^{\infty} \mathcal{A}_k = \left\{ \phi \in L^2(I):\ f = \sum_{m=0}^{\infty} a_m \psi_m,\quad \forall k,\ \|\phi\|_k < \infty \right\}.$$

The set

$$S = \left\{ \phi = \sum_{m=0}^{s} (a_m + ib_m)\, \psi_m,\ s \in \mathbb{N}_0,\ a_m, b_m \in Q \right\}$$

(Q is the set of rational numbers), is a countable dense set in each $\mathcal{A}_k$, $k \in \mathbb{N}_0$, and hence in $\mathcal{A}$. Also, since $S \subset \mathcal{A}$, $\mathcal{A}$ is dense in each $\mathcal{A}_k$, $k \in \mathbb{N}_0$. Thus $\mathcal{A}_k$, $k \in \mathbb{N}_0$, is the completition of $\mathcal{A}$ with respect to the norm $\|\cdot\|_k$.

Let $\mathcal{A}'$ ($\mathcal{A}'_k$) be the dual space of the space $\mathcal{A}$, ($\mathcal{A}'_k$) $k \in \mathbb{N}_0$.
Then we have

$$\mathcal{A}' = \bigcup_{k=0}^{\infty} \mathcal{A}_k$$

From [13, ch. 9.3. and 9.6.] it follows that

$$(\psi_m, \mathcal{R}^k \phi) = (\mathcal{R}^k \psi_m, \phi), \quad m, k \in \mathbb{N}_0, \quad \phi \in \mathcal{A}$$

where for

$$\phi \in \mathcal{A}, \quad f \in \mathcal{A}', \quad (f, \phi) = \langle f, \bar{\phi} \rangle .$$

3. Generalized random processes on $\mathcal{A}$

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. Throughout this paper we shall assume that $(\Omega, \mathcal{F}, \mathcal{P})$ is fixed.

Definition 3.1. *A generalized random processes on* $\mathcal{A}$ *is a mapping* $\xi : \Omega \times \mathcal{A} \to \mathbb{C}$ *such that*

(i) $\forall \phi \in \mathcal{A}$, $\xi(\cdot, \phi)$ *is a random variable on* Ω,

(ii) $\forall \omega \in \Omega$, $\xi(\omega, \cdot)$ *is an element from* $\mathcal{A}'$.

In [9] representation theorems for a g.r.p. on $\mathcal{A}$ were obtained. In this paper we shall need only the representation of a g.r.p. on $\mathcal{A}$ on a set $B \in \mathcal{F}$ with arbitrary large probability.

Theorem 3.1. *Let* ξ *be a g.r.p. on* $\mathcal{A}$. *Then for every* $\varepsilon > 0$ *there exist a set* $B \in \mathcal{F}$, *with* $P(B) \geq 1-\varepsilon$, *an integer* $k_0 = k_0(\varepsilon) \in \mathbb{N}_0$, *and a sequence of random variables on* Ω, $\{c_m,\ m \in \mathbb{N}_0\}$ *such that*

$$\xi(\omega, \phi) = \sum_{m=0}^{\infty} c_m(\omega) (\psi_m, \phi), \quad \omega \in B, \quad \phi \in \mathcal{A} \tag{3.1}$$

and

$$(3.2) \qquad \Big[\sum_{m=0}^{\infty} |c_m(\omega)|^2 \, \tilde{\lambda}_m^{-2k_0} \Big]^{1/2} < k_0 , \quad \omega \in B .$$

The proof is given in [9, Theorem 3.1.]. See also [1,11,13] .

We define the differential operator $(\mathcal{R}')^k$, $k \in \mathbb{N}_0$ on the set of g.r.p.-s by

$$(\mathcal{R}')^k \, \xi(\omega,\phi) = \xi\,(\omega,\mathcal{R}^k\phi), \quad \omega \in \Omega, \quad \phi \in \mathcal{A} .$$

$$(\mathcal{R}')^{k+1} = \mathcal{R}'((\mathcal{R}')^k), \quad k \in \mathbb{N}_0 \qquad (\mathcal{R}')^0 = \mathcal{J}$$

We shall denote $\mathcal{R}'$ by $\mathcal{R}$. Put $\Lambda = \{n \in \mathbb{N}_0 : \lambda_n = 0\}$, $\Lambda^c = \mathbb{N}_0 \setminus \Lambda$.

Theorem 3.2. *Let ξ be a g.r.p. on $\mathcal{A}$. For every $\varepsilon > 0$ there exist a set $B \in \mathcal{F}$ with $P(B) \geq 1-\varepsilon$, an integer $k_0 = k_0(\varepsilon)$, a function $X_k : \Omega \times I \to \mathbb{C}$ and random variables $\{c_m ,\ m \in \mathbb{N}_0\}$ such that for every $k \geq k_0$.*

$$(3.3) \qquad \xi(\omega,\phi) = \int_I X_k(\omega,t)\, \mathcal{R}^k\phi(t)dt + \sum_{m\in\Lambda} c_m(\omega)(\psi_m,\phi), \quad \omega \in B, \quad \phi \in \mathcal{A},$$

$$(3.4) \qquad \|X_k(\omega,\cdot)\|_{L^2} < k , \quad \omega \in B .$$

The proof of (3.3) is the same as in [9, Theorem 3.3. and 3.4.]. We note only that here, we shall take X_k in the form

$$X_k(\omega,t) = \sum_{m=0}^{\infty} b_m(\omega)\psi_m(t) , \quad t \in I, \quad \omega \in \Omega ,$$

where

$$b_m(\omega) = \begin{cases} c_m(\omega)\, \tilde{\lambda}_m^{-k} , & \omega \in B \\ 0 , & \omega \notin B \end{cases} \qquad m \in \mathbb{N}_0 .$$

We have that $\|X_k(\omega,\cdot)\|_{L^2}$, $\omega \in \Omega$ is a random variable, since

$$\|X_k(\omega,\cdot)\|_{L^2} = \begin{cases} \sup \{ |\xi(\omega,\phi)|, \phi \in S_r, \ \|\phi\|_{k\leq 1} \}, & \omega \in B \\ 0 , & \omega \notin B \end{cases} =$$

$$= \begin{cases} \Big[\sum_{m=0}^{\infty} |c_m(\omega)|^2 \, \tilde{\lambda}_m^{-2k} \Big]^{1/2} , & \omega \in B \\ 0 , & \omega \notin B \end{cases} < k .$$

In [9] the following conditions were posed on sequence $\{\lambda_m,\ n \in \mathbb{N}_0\}$ and $\{\psi_m,\ m \in \mathbb{N}_0\}$ in order to obtain the representation with a continuous process X_k. By a continuous process on $\Omega \times I$ we shall mean the process that is, for almost every $\omega \in \Omega$, a continuous function on I.

(*) There exist $s_0 \in \mathbb{N}_0$ and a constant K such that, for $s \geq s_0$

$$\sup\left\{|\psi_m(t)/\tilde{\lambda}_m^s| : m \in \mathbb{N}_0,\ t \in I\right\} < K .$$

(**) There exist $p_0 \in \mathbb{N}_0$ such that for $p \geq p_0$

$$\sum_{m=0}^{\infty} |\tilde{\lambda}_m|^{-2p} < \infty .$$

Theorem 3.3. *Let ξ be a g.r.p. on $\mathcal{A}$. Then, for every $\varepsilon > 0$ there exist a set $B \in \mathcal{F}$, with $P(B) \geq 1-\varepsilon$, an integer $k_0 = k_0(\varepsilon)$, random variables c_m, $m \in \Lambda$, and a continuous random process $X_k(\omega, t)$ on $\Omega \times I$, such that for $k \geq k_0$, $p \geq p_0$, $s \geq s_0$.*

$$(3.5)\qquad \xi(\omega,\phi) = \int_I X_k(\omega,t)\, \mathcal{R}^{k+p+s}\phi(t)dt + \sum_{m\in\Lambda} c_m(\omega)(\psi_m,\phi), \quad \omega \in B, \quad \phi \in \mathcal{A},$$

$$(3.6)\qquad \|X_k(\omega,\cdot)\|_{L^2} < k , \quad \omega \in \Omega .$$

The proof is similar to the proof of Theorem 3.5., [9], where the same representation as in (3.5) was obtained on a set $A \in \mathcal{F}$, with $P(A)=0$, under an additional condition. Relation (3.6) follows in the same way as in Theorem 3.2. . Again, we note that X_k has the form

$$X_k(\omega,t) = \begin{cases} \sum_{m=0}^{\infty} c_m(\omega)\, \tilde{\lambda}_m^{-(k+p+s)} \psi_m(t) , & \omega \in B, \quad t \in I \\ 0 , & \omega \notin B, \quad t \in I . \end{cases}$$

4. Convergence of generalized random processes on $\mathcal{A}$

We shall give the definitons of different types of convergences of a sequences of g.r.p.-s on $\mathcal{A}$, following [4].

Definiton 4.1. *The sequence $\{\xi_n,\ n \in \mathbb{N}_0\} = \{\xi_n\}$ of g.r.p.-s on $\mathcal{A}$ is said to converge to the g.r.p. ξ in probability ($\mathcal{A}'$) if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}_0$ such that*

$$\lim_{n\to\infty} P\left\{\omega \in \Omega \,\middle|\, \sup_{\|\phi\|_k \le 1} |\xi_n(\omega_1,\phi) - \xi(\omega,\phi)| \ge \varepsilon \right\} = 0$$

In short, we shall write

$$\xi_n \xrightarrow{P} \xi \ (\mathcal{A}')$$

Definition 4.2. *The sequence* $\{\xi_n\}$ *of g.r.p.-s on* $\mathcal{A}$ *is said to converge to the g.r.p.* ξ *im mean* $(\mathcal{A}')$ *if there exists* $k \in \mathbb{N}_0$ *such that*

$$\lim_{n\to\infty} \int_{\Omega} \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi) - \xi(\omega,\phi)| dP(\omega) = 0$$

In short, we shall write

$$\xi_n \xrightarrow{1} \xi \ (\mathcal{A}')$$

Obviously, convergences in probability [mean] $(\mathcal{A}')$ given above imply the weak convergences in probability [mean].

Definition 4.3. *(see also* [9]*). The sequence* $\{\xi_n\}$ *of g.r.p.-s on* $\mathcal{A}$ *is said to converge to a g.r.p.* ξ *almost surely* $(\mathcal{A}')$*, if there exists a set* $Z \in \mathcal{F}$*, with* $P(Z)=0$ *and for* $\omega \in \Omega\setminus Z$, $\xi_n(\omega,\cdot) \to \xi(\omega,\cdot)$ *weakly.*

In [9] representation theorems for a sequence of g.r.p.-s on $\mathcal{A}$ converging almost surely $(\mathcal{A}')$ were obtained. To obtain representation theorems of g.r.p.-s converging in probability and mean $(\mathcal{A}')$ we need a bound condition as in B(*ii*) of Theorem 4.1. of [9]. See also [4]. Thus we give:

Definition 4.4. *The sequence* $\{\xi_n\}$ *of g.r.p.-s on* $\mathcal{A}$ *is said to converge to the g.r.p.* ξ *boundedly in probability* [mean] $(\mathcal{A}')$*, if*

(*i*) $\xi_n \xrightarrow{P} \xi \ (\mathcal{A}')$, $[\xi_n \xrightarrow{1} \xi \ (\mathcal{A}')]$

(*ii*) *there exists a set* $Z \in \mathcal{F}$*, such that* $P(Z)=0$ *and for* $\omega \in \Omega\setminus Z$ $\{\xi_n(\omega,\cdot)\}$ *is bounded in* $(\mathcal{A}')$.

In short, we shall write $\xi_n \xrightarrow[b]{P} \xi \ (\mathcal{A}')$, $[\xi_n \xrightarrow[b]{1} \xi \ (\mathcal{A}')]$.

Obviously, $\xi_n \xrightarrow[b]{P} \xi \ (\mathcal{A}') \Rightarrow \xi_n \xrightarrow{P} \xi \ (\mathcal{A}')$ $[\xi_n \xrightarrow[b]{1} \xi \ (\mathcal{A}')] \Rightarrow [\xi_n \xrightarrow{1} \xi \ (\mathcal{A}')]$.

We have that (see [4,9]) condition (*ii*) of the above definition is equivalent to *(ii′) For every* $\varepsilon > 0$ *there exists set* $B \in \mathcal{F}$, *with* $P(B) \geq 1-\varepsilon$, *an integer* $k \in \mathbb{N}_0$, *independent of* n, *such that for every* $\omega \in B$, $\phi \in \mathcal{A}$ $|\xi_n(\omega,\phi)| \leq k\|\phi\|_k$.

Since $\xi_n \to \xi$ iff $\xi_n - \xi \to 0$, we shall consider the case $\xi_n \to 0$.

Theorem 4.1. *Let* $\{\xi_n\}$ *be a sequence of g.r.p.-s on* $\mathcal{A}$. *If* $\xi_n \to 0$ *boundedly in probability [mean]* $(\mathcal{A}')$, *then for every* $\varepsilon > 0$ *there exist a set* $B \in \mathcal{F}$, *such that* $P(B) \geq 1-\varepsilon$, *an integer* $k_0 \in \mathbb{N}_0$, *independent of* n, *and for every* $n \in \mathbb{N}_0$, *a sequence* $\{c_{m,n},\ m \in \mathbb{N}_0\}$ *of random variables on* Ω, *such that*

$$\xi_n(\omega,\phi) = \sum_{m=0}^{\infty} c_{m,n}(\omega)(\psi_m,\phi), \quad \omega \in B, \quad \phi \in \mathcal{A} \tag{4.1}$$

$$\left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0} \right]^{1/2} < k_0 , \quad \omega \in B . \tag{4.2}$$

(4.3) *for each* $\delta > 0$

$$P\left\{ \omega \in B : \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0} \right]^{1/2} > \delta \right\} \to 0, \quad n \to \infty ,$$

$$\left[\int_B \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0} \right]^{1/2} dP(\omega) \to 0, \quad n \to \infty \right] ,$$

$$P\left\{ \omega \in B : |c_{m,n}(\omega)| > \delta \right\} \to 0, \quad n \to \infty , \ m \in \mathbb{N}_0 \tag{4.4}$$

Proof. Assume that $\xi_n \xrightarrow[b]{P} 0$ $(\mathcal{A}')$, $[\xi_n \xrightarrow[b]{1} 0$ $(\mathcal{A}')]$ and let $\varepsilon > 0$ given. From equivalence *(ii)* and *(ii′)*, there exist a set $B \in \mathcal{F}$, with $P(B) \geq 1-\varepsilon$ and an integer $k_0 \in \mathbb{N}_0$ such that for each $\omega \in B$ and $\phi \in \mathcal{A}$, $|\xi_n(\omega,\phi)| \leq k_0\|\phi\|_{k_0}$.
Thus, (4.1) and (4.3) follow from Theorem 3.1.

We have that (see the proof of Theorem 3.1. of [9])

$$\sup_{\|\phi\|_{k_0} \leq 1} |\xi_n(\omega,\phi)| = \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0} \right]^{1/2} , \quad \omega \in B .$$

Thus, for $\delta > 0$

$$P\left\{ \omega \in B : \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0} \right]^{1/2} \geq \delta \right\} =$$

$$P\left\{ \omega \in B : \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| \ge \delta \right\} \le$$

$$\le \; P\left\{ \omega \in \omega : \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| \ge \delta \right\} \to 0, \quad n \to \infty .$$

$$\left[\int_B \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k_0} \right]^{1/2} dP(\omega) = \int_B \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| dP(\omega) \le \right.$$

$$\left. \le \int_\Omega \sup_{\|\phi\|_{k_0} \le 1} |\xi_n(\omega,\phi)| \; dP(\omega) \to 0 , \quad n \to \infty \right] .$$

Putting $\phi=\psi_m$ in (4.1) we get that, for $m \in \mathbb{N}_0$, $\delta > 0$

$$P\left\{ \omega \in B : |c_{m,n}(\omega)| > \delta \right\} \to 0 , \quad n \to \infty .$$

To prove the converse, we need an additional assumption.

Theorem 4.2. *The sequence* $\{\xi_n\}$ *converges to zero boundedly in probability [mean]* $(\mathcal{A}')$, *if the following conditions hold.*

There exist $k \in \mathbb{N}_0$ *such that for every* $p \in \mathbb{N}$ *there exists a set* $B_p \in \mathcal{F}$, *with* $P(B) \ge 1 - \frac{1}{p}$, *such that*

(4.1') $$\xi_n(\omega,\phi) = \sum_{m=0}^{\infty} c_{m,n}(\omega)(\psi_m,\phi), \quad \omega \in B_p, \quad \phi \in \mathcal{A}$$

(4.2') $$\left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k} \right]^{1/2} < k , \quad \omega \in B_p ,$$

(4.3') *for every* $\delta > 0$

$$P\left\{ \omega \in B_p : \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k} \right]^{1/2} > \delta \right\} \longrightarrow 0, \quad n \to \infty ,$$

$$\left[\int_{B_p} \left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 \; \tilde{\lambda}_m^{-2k} \right]^{1/2} dP(\omega) \longrightarrow 0, \quad n \to \infty \right] .$$

Proof. Put $\varepsilon=1/p$. Then we have that for every $p \in \mathbb{N}$ there exists a set $B_p \in \mathcal{F}$ with $P(B_p) \ge 1 - 1/p$, such that (4.1'), (4.2'), (4.3') hold. Let $\Omega_1 = \bigcup_{p=0}^{\infty} B_p$. We have that $P(\Omega_1)=1$, and for $\omega \in \Omega$, $\phi \in \mathcal{A}$

$$|\xi_n(\omega,\phi)| = \Big[\sum_{m=0}^{\infty} c_{m,n}(\omega)|^2|\tilde{\lambda}_m|^{-2k}\Big]\|\phi\|_k \,, \le k\|\phi\|_k ,$$

thus (*ii*) is satisfied.

To prove (*i*), let $\varepsilon > 0$ be given. Then, there exists, $p \in \mathbb{N}$ such that $P(B_p) \ge 1 - \frac{\varepsilon}{2}$. Also, from (4.3') it follows that there exists $n_0 = n_0(\varepsilon,\delta)$, such that for $n \ge n_0$, $\delta > 0$

$$P\left\{\omega \in B_p : \Big[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2|\tilde{\lambda}_m|^{-2k}\Big]^{1/2} > \delta\right\} < \frac{\varepsilon}{2}$$

Since

$$\sup_{\|\phi\|_k\le 1}|\xi_n(\omega,\phi)| = \Big[\sum_{m=0}^{\infty} c_{m,n}(\omega)|^2|\tilde{\lambda}_m|^{-2k}\Big]^{1/2} , \quad \omega \in B_p$$

we have for every $\delta > 0$, and $n \ge n_0$,

$$P\left\{\omega \in \Omega_1 : \sup_{\|\phi\|_{k_0}\le 1}|\xi_n(\omega,\phi)| > \delta\right\} = P\left\{\omega \in B_p : \Big[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2|\tilde{\lambda}_m|^{-2k}\Big]^{1/2} > \delta\right\} +$$

$$+ P\left\{\omega \in B_p^c : \sup_{\|\phi\|_{k_0}\le 1}|\xi_n(\omega,\phi)| > \delta\right\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} , \quad n \ge n_0(\varepsilon,\delta).$$

[For $\varepsilon > 0$ there exist $p \in \mathbb{N}$ and $B_p \in \mathcal{F}$, $P(B_p) \ge 1 - \frac{\varepsilon}{2k}$. From (4.3') it follows that there exists $n_0 = n_0(\varepsilon)$ such that for $n \ge n_0$

$$\int_{B_p} \Big[\sum_{m=0}^{\infty} c_{m,n}(\omega)|^2|\tilde{\lambda}_m|^{-2k}\Big]^{1/2} dP(\omega) < \frac{\varepsilon}{2}$$

Hence,

$$\int_{\Omega_1} \sup_{\|\phi\|_{k_0}\le 1}|\xi_n(\omega,\phi)|dP(\omega) = \int_{B_p} \sup_{\|\phi\|_{k_0}\le 1}|\xi_n(\omega,\phi)|dP(\omega) +$$

$$\int_{B_p^c} \sup_{\|\phi\|_{k_0}\le 1}|\xi_n(\omega,\phi)|dP(\omega) \le$$

$$\le \int_{B_p} \Big[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2|\tilde{\lambda}_m|^{-2k}\Big]^{1/2} dP(\omega) + k\int_{B_p^c} dP(\omega) \le \frac{\varepsilon}{2} + k\,\frac{\varepsilon}{2k} = \varepsilon ,$$

for $n \ge n_0$.]

Theorem 4.3. *Let* $\{\xi_n\}$ *be a sequence of g.r.p.-s on* $\mathcal{A}$. *If* $\xi_n \xrightarrow[b]{P} 0$ $(\mathcal{A}')$ $[\xi_n \xrightarrow[b]{P} 0$ $(\mathcal{A}')]$ *then for every* $\varepsilon > 0$ *there exist a set* $B \in \mathcal{F}$, *with* $P(B) > 1-\varepsilon$, *an integer* $k_0 \in \mathbb{N}_0$, *both independent of* n, *for each* $m \in \Lambda$ *a sequence of random variables* $\{c_{m,n},\ n \in \mathbb{N}_0\}$, *and for every* $k \geq k_0$ *a sequence of functions* $X_{k,n}$ *of* $\Omega \times I$, *such that*

$$(4.5)\qquad \xi_n(\omega,\phi) = \int_I X_k(\omega,t)\, \mathcal{R}^k\phi(t)dt + \sum_{m\in\Lambda} c_{m,n}(\omega)(\psi_m,\phi)\ , \quad \omega \in B, \quad \phi \in \mathcal{A}\ ,$$

$$(4.6)\qquad \|X_{k,n}(\omega,\cdot)\|_{L^2} < k\ , \quad \omega \in \Omega$$

$$(4.7)\qquad \|X_{k,n}(\omega,\cdot)\|_{L^2} \xrightarrow{P} 0\ , \quad \left[\|X_{k,n}(\omega,\cdot)\|_{L^2} \xrightarrow{1} 0\right]$$

$$(4.8)\quad \text{for every}\ \ \delta > 0 \qquad P\left\{\omega \in B : \left[\sum_{m\in\Lambda} |c_{m,n}(\omega)| > \delta\right\}\right. \to 0,\ n \to \infty$$

$$\left[\int_B \sum_{m\in\Lambda} |c_{m,n}(\omega)|dP \to 0,\ n \to \infty\right]$$

Proof. From Theorem 3.2. and the equivalence of (*ii*) and (*ii'*), (4.5) follows where for $n \in \mathbb{N}_0$, $k \geq k_0$

$$X_{k,n}(\omega,t) = \begin{cases} \sum_{m=0}^{\infty} c_{m,n}(\omega)\ \tilde{\lambda}_m^{-k}\ \psi_m(t)\ , & \omega \in B, \quad t \in I \\ 0 & ,\ \omega \notin B, \quad t \in I \end{cases}$$

Thus we have,

$$\|X_{k,n}(\omega,\cdot)\|^2_{L^2} = \begin{cases} \sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2 |\tilde{\lambda}_m|^{-2k}\ , & \omega \in B, \quad t \in I \\ 0 & ,\ \omega \notin B, \quad t \in I \end{cases} < k$$

and, for $\omega \in B$, $\phi \in \mathcal{A}$

$$\|X_{k,n}(\omega,\cdot)\|_{L^2} = \sup_{\|\phi\|_k \leq 1} |\xi_n(\omega,\phi)|$$

Thus, for every $\delta > 0$

$$P\left\{\omega \notin B : \|X_{k,n}(\omega,\cdot)\|_{L^2} > \delta\right\},$$

and therefore

$$P\left\{ \omega \in \Omega : \|X_{k,n}(\omega,\cdot)\|_{L^2} > \delta \right\} =$$

$$= P\left\{ \omega \in B : \|X_{k,n}(\omega,\cdot)\|_{L^2} > \delta \right\} = P\left\{ \omega \in B : \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)| > \delta \right\} \le$$

$$\le P\left\{ \omega \in \Omega : \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)| > \delta \right\} \to 0, \quad n \to \infty.$$

Thus, $\|X_{k,n}(\omega,\cdot)\|_{L^2} \to 0$.

$$\left[\int_\Omega \|X_{k,n}(\omega,\cdot)\|_{L^2} dP(\omega) = \int_B \|X_{k,n}(\omega,\cdot)\|_{L^2} dP(\omega) = \right.$$

$$\left. = \int_B \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)|dP(\omega) \le \int_\Omega \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)|dP(\omega) \to 0 \right].$$

Hence (4.7) follows.

For $m \in \Lambda$ put $\phi=\psi_m$ in (4.5) and we get $P\left\{ \omega \in B \,|\, |c_{m,n}(\omega)| > \delta \right\} \to 0$ $m \in \Lambda$. Since Λ is finite, (4,8) follows.

Theorem 4.4. *$\xi_n \xrightarrow[b]{P} 0$ $(\mathcal{A}')$ $[\xi_n \xrightarrow[b]{1} 0$ $(\mathcal{A}')]$ if there exist $k \in \mathbb{N}_0$ such that for every $p \in \mathbb{N}$ there exist $B_p \in \mathcal{F}$ with $P(B_p) \ge 1 - 1/p$ such that*

(4.5') $\xi_n(\omega,\phi) = \int_I X_k(\omega,t)\, \mathcal{R}^k f(t)dt + \sum_{m\in\Lambda} c_{m,n}(\omega)(\psi_m,\phi)$, $\omega \in B_p$, $\phi \in \mathcal{A}$,

(4.6') $\|X_{k,n}(\omega,\cdot)\|_{L^2} < k$, $\omega \in \Omega$

(4.7') $\|X_{k,n}(\omega,\cdot)\|_{L^2} \xrightarrow{P} 0$, $n \to \infty$, $\left[\|X_{k,n}(\omega,\cdot)\|_{L^2} \xrightarrow{1} 0, \ n \to \infty \right]$.

(4.8) for every $\delta > 0$ $P\left\{\omega \in B_p : \left[\sum_{m\in\Lambda} |c_{m,n}(\omega)| > \delta \right\} \to 0, \ n \to \infty\right.$

$$\left[\int_B \sum_{m\in\Lambda} |c_{m,n}(\omega)|dP \to 0, \ n \to \infty \right].$$

The proof is the same as the proof of Theorem 4.2.

Since

$$\|X_{k,n}(\omega,\cdot)\|_{L^2} = \begin{cases} \sup\limits_{\|\phi\|_k\le 1} |\xi_n(\omega,\phi)| , & \omega \in B_p \\ 0 , & \omega \notin B_p . \end{cases}$$

Suppose that (*) and (**) are satisfied.

Theorem 4.5. *Let* $\{\xi_n\}$ *be a sequence of g.r.p.-s on* $\mathcal{A}$. *If* $\xi_n \xrightarrow[b]{P} 0\ (\mathcal{A}')$ $[\xi_n \xrightarrow[b]{1} 0\ (\mathcal{A}')]$, *then for every* $\varepsilon > 0$ *there exist a set* $B \in \mathcal{F}$ *with* $P(B) \ge 1-\varepsilon$, *an integer* $k_0 \in \mathbb{N}_0$, *both independent of* n, *for each* $m \in \Lambda$ *a sequence of random variables* $\{c_{m,n}(\omega),\ n \in \mathbb{N}_0\}$, *and for every* $k \ge k_0$ *a sequence of continuous random processes* $X_{k,n}$ *on* $\Omega \times I$, *such that, for* $n \in \mathbb{N}_0$

(4.9) $\xi_n(\omega,\phi) = \int_I X_{k,n}(\omega,t)\mathcal{R}^{k+p+m}\phi(t)dt + \sum_{m\in\Lambda} c_{m,n}(\omega)(\psi_m,\phi),\ \omega \in B,\ \phi \in \mathcal{A}$,

where $s \ge s_0$, $p \ge p_0$.

(4.10) $$\|X_{k,n}(\omega,\cdot)\|_{L^2} < k , \quad \omega \in \Omega ,$$

(4.11) $\{X_{k,n}(\omega,\cdot)\}$ *is equicontinuous on* I, *for* $p > p_0$,

(4.12) *for each* $t \in I$, *and* $k > k_0$ $X_{k,n}(\cdot,t) \xrightarrow{P} 0$, $n \to 0$,

$$\left[X_{k,n}(\cdot,t) \xrightarrow{1} 0,\ n \to \infty\right]$$

(4.13) for every $\delta > 0$ $P\left\{\omega \in B_p : |\sum_{m\in\Lambda} |c_{m,n}(\omega)| > \delta\right\} \to 0,\ n \to \infty$

$$\left[\int_B |\sum_{m\in\Lambda} |c_{m,n}(\omega)| dP \to 0,\ n \to \infty\right] .$$

Proof. From Theorem 3.3. and equivalence of *(ii)* and *(ii')* (4.9) follow, where for $n \in \mathbb{N}_0$, $k \ge k_0$

$$X_{k,n}(\omega,t) = \begin{cases} \sum\limits_{m=0}^{\infty} c_{m,n}(\omega)\ \tilde{\lambda}_m^{-(k+p+m)}\psi_m(t) , & \omega \in B,\quad t \in I \\ 0 , & \omega \notin B,\quad t \in I \end{cases}$$

(4.10) and (4.13) follow in the same manner as in Theorem 4.3. The proof of (4.11) is given in Theorem 4.3. of [9].

To prove (4.12) it is enough to see that

$$|X_{k,n}(\omega,t)| \le \sum_{m=0}^{\infty} |c_{m,n}(\omega)|\,|\tilde{\lambda}_m|^{-k}\,|\psi_m(t)|\,|\tilde{\lambda}_m|^{-s}|\tilde{\lambda}_m|^{-p}$$

$$\le K^2\, C\left[\sum_{m=0}^{\infty} |c_{m,n}(\omega)|^2\ |\tilde{\lambda}_m|^{-2k}\right]^{1/2}, \quad \omega \in B\,,$$

where $\sum_{m=0}^{\infty}|\tilde{\lambda}_m|^{-2p} = C$. Since $X_{k,n}(\omega,t) = 0$, $\omega \notin B$ we have, for every $\delta > 0$, and $t_0 \in I$

$$P\left\{\omega \in \Omega \;|\; |X_{k,n}(\omega,t_0)| > \delta\right\} \le P\left\{\omega \in B : \left[\sum_{m=0}^{\infty}|c_{m,n}(\omega)|^2|\tilde{\lambda}_m|^{-2k}\right]^{1/2} > \delta\right\} \le$$

$$\le P\left\{\omega \in \Omega : \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)| > \delta\right\} \to 0, \quad n \to \infty\,.$$

$$\left[\int_\Omega |X_{k,n}(\omega,t_0)|\,dP(\omega) = \int_B |X_{k,n}(\omega,t_0)|\,dP(\omega) = \int_B\left[\sum_{m=0}^{\infty}|c_{m,n}(\omega)|^2|\tilde{\lambda}_m|^{-2k}\right]^{1/2} dP(\omega) \le\right.$$

$$\le \int_\Omega \sup_{\|\phi\|_k \le 1} |\xi_n(\omega,\phi)|\ dP(\omega) \to 0, \quad n \to \infty\,.$$

The converse of Theorem 4.5. is not true.

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Rezime

KONVERGENCIJA NIZA UOPŠTENIH SLUČAJNIH PROCESA NA ZEMANIANOVOM PROSTORU $\mathcal{A}$

Definisane su i uporedene različite vrste konvergencija niza uopštenih slučajnih procesa na Zemanianovom prostoru $\mathcal{A}$.

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