Univ. u Novom Sadu,
Zb. Rad. Prirod.-Mat. Fak.
Ser. Mat. 19,2, 233-240 (1989)

REVIEW OF RESEARCH
FACULTY OF SCIENCE
MATHEMATICS SERIES

ON COINCIDENCE POINTS IN CONVEX METRIC SPACES

Olga Hadžić

University of Novi Sad, Faculty of Science,
Institute of Mathematics
Dr Ilije Djuričića 4, 21000 Novi Sad, Yugoslavia

ABSTRACT

In this paper a theorem on coincidence points for f, S and T is proved, where f is a multivalued mapping and S and T are singlevalued mappings. The obtained theorem generalizes Theorem 1 from [3].

1. INTRODUCTION

Many authors proved fixed point theorems or theorems on coincidence points in convex metric spaces [1], [2], [3], [4], [6], [7], [9], [10].

Let us recall that a metric space (M,d) is convex if for each $x,y \in M$ with $x \neq y$ there exists $z \in M$, $x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

In [3] we introduced the notion of a weakly commutative pair (f,S) where f is a multivalued and S is a singlevalued mapping in the following way, where $d(a,B) = \inf_{b \in B} d(a,b)$ and $a \in M$, $B \subseteq M$.

Definition 1. *Let (M,d) be a metric space, K a nonempty sub-*

AMS Mathematics Subject Classification (1980): 54H25.

Key words and phrases: Coincidence points, multivalued mappings, convex metric spaces, metric spaces with a convex structure.

set of M, CB(M) *the family of all nonempty, closed and bounded subsets of* M, $f : K \to CB(M)$ *and* $S : K \to M$. *The pair* (f,S) *is said to be weakly commutative if and only if for every* $y \in K$ *and* $z \in K$ *such that* $y \in fz$ *and* $Sz \in K$, *the following inequality holds*

$$d(Sy, fSz) \leq d(fz, Sz).$$

For singlevalued mappings f and S the notion of a weakly commutative pair is introduced by Sessa in [8]. There are examples of mappings which are weakly commutative but not commutative.

We shall give the following generalization of the notion of a weakly commutative pair (f,S) where f is a multivalued and S is a singlevalued mapping.

Definition 2. *Let* (M,d) *be a metric space,* K *a nonempty subset of* M, $f : K \to CB(M)$ *and* $S : K \to M$. *The pair* (f,S) *is said to be compatible if for every sequence* $\{x_n\}_{n \in \mathbb{N}}$ *from* K *from the relations*

$$\lim_{n\to\infty} d(Sx_n, fx_n) = 0 \text{ and } Sx_n \in K,\ n \in \mathbb{N}$$

it follows that

$$\lim_{n\to\infty} d(Sy_n, fSx_n) = 0$$

for every sequence $\{y_n\}_{n \in \mathbb{N}}$ *from* K *such that* $y_n \in fx_n$, $n \in \mathbb{N}$.

For singlevalued mappings S and f the notion of the compatibility is introduced by Jungck [5]. It is obvious that a weakly commutative pair (f,S) is also a compatible one. There are examples of compatible pairs which are not weakly commutative.

In [3] the following result is obtained, where H denotes the Hausdorff metric.

Theorem A. *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, $S, T : K \to M$ *continuous mappings,* $f : K \to CB(M)$ H-*continuous mapping,* $\partial K \subseteq SK \cap TK$, $fK \cap K \subseteq SK \cap TK$, (f,S) *and* (f,T) *weakly commutative pairs and the following implications hold:*

$$Tx \in \partial K \Rightarrow fx \in K; \quad Sx \in \partial K \Rightarrow fx \in K.$$

If there exists $q \in (0,1)$ *so that*

$$H(fx,fy) \leq qd(Sx,Ty), \quad \textit{for every } x,y \in K$$

then there exists $z \in K$ *so that* $\{Tz,Sz\} \cap fz \neq \emptyset$.

If $S,T : M \to M$ *are continuous then there exists* $z \in K$ *such that* $Tz \in fz$ and $Sz \in fz$.

Remark. If $S : M \to M$, we suppose in Definition 1 that $y \in fz$ implies $d(Sy,fSz) \leq d(fz,Sz)$ for every $z \in K$ such that $Sz \in K$.

In this paper we shall prove a generalization of Theorem A using the notion of a compatible pair and a result from [7] stated here as Theorem B.

In Theorem B, $\mathbb{R}^+$ stands for the nonnegative reals.

Theorem B. *Let* $c : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function such that* $c(t^+) < t$ *for all* $t > 0$ *and* $\sum_n c^n(t)$ *is finite for all* $t > 0$. *Then, there exists a strictly increasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $c(t) < \psi(t) < t$, *for all* $t > 0$ *and* $\sum_n \psi^n(t)$ *is finite for* $t > 0$.

Theorem. *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, S, $T : K \to M$ *continuous mappings,* $f : K \to CB(M)$ *H-continuous mapping,* $\partial K \subseteq SK \cap TK$, $fK \cap K \subseteq SK \cap TK$, (f,S) *and* (f,T) *compatible pairs and the following implications holds:*

$$Tx \in \partial K \Rightarrow fx \in K; \quad Sx \in \partial K \Rightarrow fx \in K.$$

If there exists an increasing function $c : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $c(t^+) < t$, *for all* $t > 0$ *and* $\sum_n c^n(t)$ *is finite, for all* $t > 0$ *so that*

$$H(fx,fy) \leq c(d(Sx,Ty)), \quad \textit{for every } x,y \in K,$$

then there exists $z \in K$ *so that*

$$\{Tz,Sz\} \cap fz \neq \emptyset.$$

If $S,T : M \to M$ *are continuous then there exists* $z \in K$ *so that* $Tz \in fz$, $Sz \in fz$.

Proof. As in [3] let $x \in \partial K$ and $p_0 \in K$ such that $x = Tp_0$. From $Tp_0 \in \partial K$ it follows that $fp_0 \in K \cap fK \subseteq SK$. Hence, there exists $p_1 \in K$ such that $Sp_1 \in fp_0 \subseteq K$ and let $Sp_1 = p_1'$. Further

$$d(p_1',fp_1) \leq H(fp_0,fp_1) \leq c(d(Sp_1,Tp_0))$$

and if $d(Sp_1,Tp_0) > 0$ from Theorem B we conclude that

$$d(p_1',fp_1) < \psi(d(Sp_1,Tp_0)).$$

So, there exists $p_2' \in fp_1$ such that

$$d(p_1',p_2') \leq \psi(d(Sp_1,Tp_0)). \tag{1}$$

Suppose that $d(Sp_1,Tp_0) = 0$. Then $c(d(Sp_1,Tp_0)) = H(fp_0,fp_1) = 0$ and if we take that $p_2' = p_1'$ we obtain that (1) holds. If $p_2' \in K$ then $p_2' \in K \cap fK \subseteq TK$ and so there exists $p_2 \in K$ such that $Tp_2 = p_2'$. If $p_2' \notin K$ then there exists $p_2 \in K$ such that

$$d(Sp_1,Tp_2) + d(Tp_2,p_2') = d(Sp_1,p_2').$$

Then $d(p_2',fp_2) \leq H(fp_1,fp_2) \leq c(d(Sp_1,Tp_2))$ and if $d(Sp_1,Tp_2) > 0$ it follows that

$$d(p_2',fp_2) < \psi(d(Sp_1,Tp_2))$$

which implies that there exists $p_3' \in fp_2$ such that

$$d(p_2',p_3') \leq \psi(d(Sp_1,Tp_2)). \tag{2}$$

If $d(Sp_1,Tp_2) = 0$ we take that $p_3' = p_2'$ and so (2) holds.

In this way we obtain two sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{p_n'\}_{n\in\mathbb{N}}$ such that:

1. For every $n \in \mathbb{N}$: $p_n \in fp_{n-1}$.
2. For every $n \in \mathbb{N}$: $p_{2n}' \in K \Rightarrow p_{2n}' = Tp_{2n}$;

   $p_{2n}' \notin K \Rightarrow Tp_{2n} \in \partial K$ and

$$d(Sp_{2n-1},Tp_{2n}) + d(Tp_{2n},p_{2n}') = d(Sp_{2n-1},p_{2n}'). \tag{3}$$

3. For every $n \in \mathbb{N}$: $p'_{2n+1} \in K \Rightarrow p'_{2n+1} = Sp_{2n+1}$; $p'_{2n+1} \notin K \Rightarrow Sp_{2n+1} \in \partial K$ and

$$d(Tp_{2n},Sp_{2n+1}) + d(Sp_{2n+1},p'_{2n+1}) = d(Tp_{2n},p'_{2n+1}). \tag{4}$$

4. For every $n \in \mathbb{N}$:

$$d(p'_{2n},p'_{2n+1}) \le \psi(d(Sp_{2n-1},Tp_{2n})),$$

$$d(p'_{2n+1},p'_{2n+2}) \le \psi(d(Sp_{2n+1},Tp_{2n})).$$

Let P_0, P_1, Q_0 and Q_1 be defined by

$$P_0 = \{p_{2n},\ n \in \mathbb{N} \text{ and } p'_{2n} = Tp_{2n}\},$$

$$P_1 = \{p_{2n},\ n \in \mathbb{N} \text{ and } p'_{2n} \ne Tp_{2n}\},$$

$$Q_0 = \{p_{2n+1},\ n \in \mathbb{N} \text{ and } p'_{2n+1} = Sp_{2n+1}\},$$

$$Q_1 = \{p_{2n+1},\ n \in \mathbb{N} \text{ and } p'_{2n+1} \ne Sp_{2n+1}\}.$$

It is easy to prove that

$$(p_{2n},p_{2n+1}) \notin P_1 \times Q_1, \quad (p_{2n-1},p_{2n}) \notin Q_1 \times P_1.$$

If $x_{2n} = Tp_{2n}$ and $x_{2n+1} = Sp_{2n+1}$, $n \in \mathbb{N}$ we shall prove that

$$d(x_n,x_{n+1}) \le \begin{cases} \psi(d(x_{n-1},x_n)), p'_n \in K \\ \psi(d(x_{n-2},x_{n-1})),\ n \ge 2,\ p'_n \notin K. \end{cases} \tag{5}$$

1. $(p_{2n},p_{2n+1}) \in P_0 \times Q_0$;

Then we have that

$$d(Tp_{2n},Sp_{2n+1}) = d(p'_{2n},p'_{2n+1}) \le \psi(d(Sp_{2n-1},Tp_{2n}))$$

which means that

$$d(x_{2n},x_{2n+1}) \le \psi(d(x_{2n-1},x_{2n})).$$

2. $(p_{2n},p_{2n+1}) \in P_0 \times Q_1$;

Then from (4) we have that

$$d(Tp_{2n},Sp_{2n+1}) \le d(Tp_{2n},p'_{2n+1}) = d(p'_{2n},p'_{2n+1}) \le \psi(d(Sp_{2n-1},Tp_{2n}))$$

as in case 1.

3. $(p_{2n},p_{2n+1}) \in P_1 \times Q_0$;

Then

$$d(Tp_{2n},Sp_{2n+1}) \le d(Tp_{2n},p'_{2n}) + d(p'_{2n},p'_{2n+1}) \le$$

$$\le d(Tp_{2n},p'_{2n}) + \psi(d(Sp_{2n-1},Tp_{2n})) \le d(Tp_{2n},p'_{2n}) +$$

$$+ d(Sp_{2n-1},Tp_{2n})$$

and from (3) we obtain that

$$d(Tp_{2n},Sp_{2n+1}) \le d(Sp_{2n-1},p'_{2n}).$$

Since $p_{2n+1} \in Q_0$ we have $p'_{2n-1} = Sp_{2n-1}$. This implies that

$$d(Tp_{2n},Sp_{2n+1}) \le d(p'_{2n-1},p'_{2n}) \le \psi(d(Sp_{2n-1},Tp_{2n-2}))$$

and so

$$d(x_{2n},x_{2n+1}) \le \psi(d(x_{2n-1},x_{2n-2})).$$

4. $(p_{2n-1},p_{2n}) \in Q_1 \times P_0$;

Then

$$d(Sp_{2n-1},Tp_{2n}) \le d(Sp_{2n-1},p'_{2n-1}) + d(p'_{2n-1},Tp_{2n}) =$$

$$= d(Sp_{2n-1},p'_{2n-1}) + d(p'_{2n-1},p'_{2n}) \le$$

$$\le d(Sp_{2n-1},p'_{2n-1}) + \psi(d(Sp_{2n-1},Tp_{2n-2})) \le$$

$$\le d(Sp_{2n-1},p'_{2n-1}) + d(Sp_{2n-1},Tp_{2n-2})$$

and since $p_{2n-1} \in Q_1$ we obtain that $d(Sp_{2n-1},Tp_{2n}) \le d(Tp_{2n-2},p'_{2n-1})$.

From $p_{2n-1} \in Q_1$ it follows that $p_{2n-2} \in P_0$ and so $Tp_{2n-2} = p'_{2n-2}$. Hence

$$d(Sp_{2n-1},Tp_{2n}) \le d(p'_{2n-2},p'_{2n-1}) \le \psi(d(Sp_{2n-3},Tp_{2n-2}))$$

and so

$$d(x_{2n-1},x_{2n}) \le \psi(d(x_{2n-3},x_{2n-2})).$$

Using 1., 2., 3. and 4. we conclude that (5) is proved. It can be proved that (5) implies that for every $n \in \mathbb{N}$

$$d(x_n,x_{n+1}) \le \psi^{k(n)}(d(x_0,x_1))$$

where

$$k(n) = \begin{cases} 1, & n = 1 \\ [n/2], & n \ge 2. \end{cases}$$

Since $\sum_n \psi^n(t)$ is finite for $t > 0$ we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and so there exists $z \in K$ such that

$$z = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1}.$$

As in [3], suppose that there exists a subsequence $\{p_{2n_k}\}_{k\in\mathbb{N}}$ such that $p_{2n_k} \in P_0$ for every $k \in \mathbb{N}$ which means that $Tp_{2n_k} \in fp_{2n_k-1}$, $k \in \mathbb{N}$. We shall prove that $Sz \in fz$ using the compatibility of (f,S). Since $Tp_{2n_k} \in fp_{2n_k-1} \cap K$, $Sp_{2n_k-1} \in K$ and

$$d(fp_{2n_k-1},Sp_{2n_k-1}) \le d(Tp_{2n_k},Sp_{2n_k-1})$$

we obtain that $\lim_{k\to\infty} d(fp_{2n_k-1},Sp_{2n_k-1}) = 0$ and from the compatibility of (f,S) it follows that $\lim_{k\to\infty} d(STp_{2n_k},fSp_{2n_k-1}) = 0$. As in [3] it follows that $Sz \in fz$ and so $\{Tz,Sz\} \cap fz \ne \emptyset$. The rest of the proof is similar to that in [3].

Corollary [7]. *Let* (X,d) *be a complete convex metric space,* K *a nonempty, closed subset of* X *and* $S : K \to CB(X)$ *be such that* $Sx \subseteq K$ *for every* $x \in \partial K$ *and*

$$H(Sx,Sy) \le c(d(x,y)), \text{ for all } x,y \in K$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ *is as in the Theorem. Then* S *has a fixed point.*

Acknowledgement. This work was partly supported by the NSF (U.S.A.) and Fund for Science of Vojvodina through funds made available to the U.S.-Yugoslav Joint Board on Scientific and Technological Cooperation, Grant JF 799.

REFERENCES

[1] N.A. Assad, W.A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math., Vol. 43, No. 3 (1972), 553-562.

[2] O. Hadžić, *On coincidence points in metric and probabilistic metric spaces with a convex structure*, Univ. u Novom Sadu, Zb. Rad. Prir.-Mat. Fak., Ser. Mat., 15,1 (1985), 11-22.

[3] O. Hadžić, Lj. Gajić, *Coincidence points for set-valued mappings in convex metric spaces*, Univ. u Novom Sadu, Zb. Rad. Prir.-Mat. Fak., Ser. Mat., 16,1 (1986), 13-25.

[4] S. Itoh, *Multivalued generalized contractions and fixed point theorems*, Comm. Math. Univ. Carolinae, 18(2) (1977), 247-258.

[5] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., 9(4) (1986), 771-779.

[6] M.S. Khan, *Common fixed point theorems for multivalued mappings*, Pacific J. Math., 95(2) (1981), 337-347.

[7] K.P.R. Sastry, S.V.R. Naidu and J.R. Prasad, *Common fixed points for multimaps in a metric space*, Nonlinear Analysis, 13,3 (1989), 221-229.

[8] S. Sessa, *On a weak commutativity condition of mappings in fixed point consideration*, Publ. Inst. Math. (Beograd), 32(46) (1982), 149-153.

[9] W. Takahashi, *A convexity in metric space and nonexpansive mappings I*, Kodai Math. Sem. Rep., 22 (1970), 142-149.

[10] L. Talman, *Fixed points for condensing multifunctions in metric spaces with convex structure*, Kodai Math. Sem. Rep., 29 (1977), 62-70.

REZIME

O TAČKAMA KOINCIDENCIJE U KONVEKSNIM METRIČKIM PROSTORIMA

U radu je dokazana teorema o tačkama koincidencije za f, S i T, gde je f višeznačno preslikavanje a S i T jednoznačna preslikavanja. Dobijena teorema uopštava teoremu 1 iz rada [3].

Received by the editors August 15 1989.