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ON AN ITERATIVE METHOD FOR A SYSTEM OF EQUATIONS

Dragoslav Herceg and **Ljiljana Cvetković**
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In order to solve systems of linear equations a modification of the Accelerated Overrelaxation (AOR) method is used, introduced in [8]. By using a technique similar to Sassenfeld's criteria, criteria of convergence are obtained which do not demand additional properties of the matrix of the linear system.

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1. Introduction

There are a lot of iterative methods for solving systems of linear equations. Among them, the so-called relaxation methods play a very important role. The iteration matrices of these methods depend on several real parameters, so it is possible to accelerate the rate of convergence by using a proper choice of these parameters. For example, some well known methods of such a type are the Jacobi, Gauss-Seidel, SOR and AOR method.

In this paper we shall consider a more general method, introduced by James, [8] . As we shall see, it is a special case of the MAOR (Modified Accelerated Overrelaxation) method, which we have introduced for the nonlinear case in paper [2] and investigated in [3],[4].

We shal obtain criteria of convergence, which do not demand additional properties of the matrix of our linear system (such as SDD or something else).

2. Preliminaries

In order to solve a system of linear equations

$$Ax = b, \tag{1}$$

where $A = [a_{ij}] \in C^{n,n}$, $b \in C^n$, $a_{ii} \neq 0$, $i \in N := \{1, 2, ..., n\}$, we shall use the following iterative method

$$x^{k+1} = M(\sigma, \Omega)x^k + d, \quad k = 0, 1, ..., \tag{2}$$

where

$$M(\sigma, \Omega) = (E - \sigma\Omega L)^{-1}((E - \Omega) + (1 - \sigma)\Omega L + \Omega U),$$

$$d = (E - \sigma\Omega L)^{-1}\Omega D^{-1}b,$$

$A = D(E - L - U)$ is the standard splitting, σ is a real parameter and $\Omega = diag(\omega_1, \omega_2, ..., \omega_n)$ is a real regular diagonal relaxation matrix.

If we choose $\omega_1 = \omega_2 = ... = \omega_n = \omega$, we obtain the AOR method with the parameters $\omega\sigma$ and ω, and if, in addition, $\sigma = 1$, the method reduced to the SOR method with the parameter ω.

Also, we can compare this method to the MAOR one. It is obvious that

$$M(\sigma, \Omega) = H(\Omega^{-1}; \sigma, 1),$$

where H denotes the MAOR iteration matrix (cf.[2], [3], [4]).

3. An upper bound for the maximum norm of iteration matrix

Theorem 1. *Let*

$$p_i = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|}(|1 - \sigma\omega_j| + |\sigma||\omega_j|p_j) + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|}, \; i \in N.$$

Then

$$\|M(\sigma, \Omega)\|_\infty \leq \max_{i \in N}\{|1 - \omega_i| + |\omega_i|p_i\}.$$

Proof. Let y be a vector with the following property:

$$\|y\|_\infty = 1, \quad \|M(\sigma,\Omega)\|_\infty = \|M(\sigma,\Omega)y\|_\infty.$$

If we denote $M(\sigma,\Omega)y = z$, we shall obtain

$$(E - \sigma\Omega L)z = ((E-\Omega) + (1-\sigma)\Omega L + \Omega U)\, y,$$

or, written by components

$$z_i + \sigma\omega_i \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} z_j = y_i - \omega_i y_i - (1-\sigma)\omega_i \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} y_j - \omega_i \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} y_j.$$

From the last equality we obtain

$$\text{(3)} \quad z_i - (1-\omega_i)y_i = -\omega_i \left[\sigma \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} z_j + (1-\sigma) \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} y_j + \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} y_j \right],$$

from which, by using mathematical induction, we can prove that

$$\text{(4)} \qquad |z_i - (1-\omega_i)y_i| \le |\omega_i| p_i, \; i \in N.$$

Indeed, for $i = 1$ it is obvious that

$$|z_1 - (1-\omega_1)y_1| \le |\omega_1| p_1.$$

Now, we suppose that

$$\text{(5)} \qquad |z_k - (1-\omega_k)y_k| \le |\omega_k| p_k, \quad k = 1,2,\ldots,i-1.$$

From (3) we obtain

$$|z_i - (1-\omega_i)y_i| \le |\omega_i| \left[\sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{ii}|} |\sigma z_j + (1-\sigma)y_j| + \sum_{j=i+1}^{n} \frac{|a_{ij}|}{|a_{ii}|} \right],$$

and it is sufficient to show that

$$|\sigma z_j + (1-\sigma)y_j| \le |\sigma||\omega_j| p_j + |1 - \sigma\omega_j|, \quad j = 1,2,\ldots,i-1.$$

Because of (5), it follows that for $j = 1,2,\ldots,i-1$

$$|\sigma z_j + (1-\sigma)y_j| = |\sigma\,(z_j - (1-\omega_j)y_j) + (1-\sigma\omega_j)y_j|$$

$$\leq |\sigma||z_j - (1-\omega_j)y_j| + |1-\sigma\omega_j|$$
$$\leq |\sigma||\omega_j|p_j + |1-\sigma\omega_j|.$$

Hence, relation (4) is proved. Now, it is obvious that

$$|z_i| \leq |1-\omega_i| + |\omega_i|p_i, \ i \in N,$$

which completes the proof.

□

The result of Theorem 1 is a generalization of the Sassenfeld's criteria. One can see that Theorem 1 gives a sufficient condition for the convergence of method (2) without extra properties of matrix A. It is sufficient to choose a parameter σ and a matrix Ω, so that

$$\max_{i\in N}\{|1-\omega_i| + |\omega_i|p_i\} < 1.$$

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REZIME

O JEDNOM ITERATIVNOM POSTUPKU ZA SISTEM JEDNAČINA

Za rešavanje sistema linearnih jednačina koristi se jedna modifikacija AOR postupka, definisana u [8]. Korišćenjem tehnike slične Sassenfeldovom kriterijumu, dobijeno je jedno gornje ograničenje za maksimum normu iterativne matrice i kriterijum za konvergenciju, koji ne zahteva dodatne pretpostavke o matrici posmatranog linearnog sistema (SDD ili nešto drugo).

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