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# AN ACCELERATION PROCEDURE OF THE WHITTAKER METHOD BY MEANS OF CONVEXITY

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**Abstract**

The paper studies the influence the convexity of a real function $f$ has in the Whittaker method, in order to get the solution of $f(x)=0$, and obtains a method for accelerating this iterative process. Also, some new iterative processes are obtained.

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*Key words and phrases:* Nonlinear equations, iterative method, convexity.

## 1. Introduction

The aim in this paper is to study the influence of the convexity in the solution of $f(x) = 0, x \in [a,b] \subset \mathbf{R}$, by the iteration $x_n = x_{n-1} - \phi_f f(x_{n-1})$, where $\phi_f$ is a constant, [1]-[6], in order to get accelerations of this method by means of the convexity of $f$. Moreover, we shall obtain new iterative processes of the second and third order, and also give sufficient conditions for their convergence.

Assume that the function $f$ satisfies the conditions

$$f(a) < 0 < f(b), f'(x) > 0, f''(x) \geq 0 \ \text{ for } x \in [a,b]$$

These conditions imply that $f \in C^{(m)}([a,b])$, $m \geq 2$, and it has one and only one root $s \in (a,b)$. Besides, we consider $x_0 \in [a,b]$ such that $f(x_0) > 0$.

First, we are going to introduce the log-degree of convexity, which will provide us with an index of convexity measure of a function at each point. This index is given by $U[f](x) = f''(x) \cdot [f'(x)]^{-2}$.

We shall prove that when the log-degree of convexity of $f, U[f](x)$, decreases, the sequence $\{x_n\}$ converges faster to $s$. Since the straight lines have a minimum log-degree of convexity, we take $g(x) = f(x) - [f''(s)/2!](x-s)^2$ as an approximation to the tangent line of $f$ at $s$. Hence, $U[g](x)$ tends to zero as $x$ tends to $s$; therefore, if we asssume $f'''(x) \geq 0$ in $(a,\ b)$, there exists an interval $(\alpha, \beta) \subset [a,b], f(\alpha) < 0 < f(\beta)$, such that $g$ is an increasing convex function and $U[g](x) < U[f](x)$ in $(\alpha, \beta)$, and it is obvious that $s$ is the unique root of $g$ in $(a,\ b)$. Therefore, when we take accelerations, we shall suppose $m \geq 3$ and $f'''(x) \geq 0$ in $[a,\ b]$.

Now, we shall consider Whittaker's method for $\phi_f = [f'(b)]^{-1}$. Then,

$$f''(s)(x_{n-1} - s)^2 \sim f''(x_{n-1})(x_{n-1} - x_n)^2 [\frac{f'(b)}{f'(x_{n-1})}]^2$$

since

$$\lim_n \frac{f''(s)(x_{n-1} - s)^2}{f''(x_{n-1})(x_{n-1} - x_n)^2 [\frac{f'(b)}{f'(x_{n-1})}]^2} = 1,$$

and it follows that

$$g(x_{n-1}) = f(x_{n-1}) - \frac{f''(x_{n-1}) f(x_{n-1})}{2! [f'(x_{n-1})]^2}$$

Then, we obtain $y_n = x_{n-1} - \phi_g g(x_{n-1})$, and this sequence converges faster to $s$ than does $\{x_n\}$, and this acceleration allows us to define new iterative processes.

## 2. An index of convexity measure

Let $h \in C^{(2)}(V)$ a convex function and $V$ a neighbourhood of a suitable $x_0 \in [a,b]$. The curvature $K$, [3], is a measure of the convexity of a function

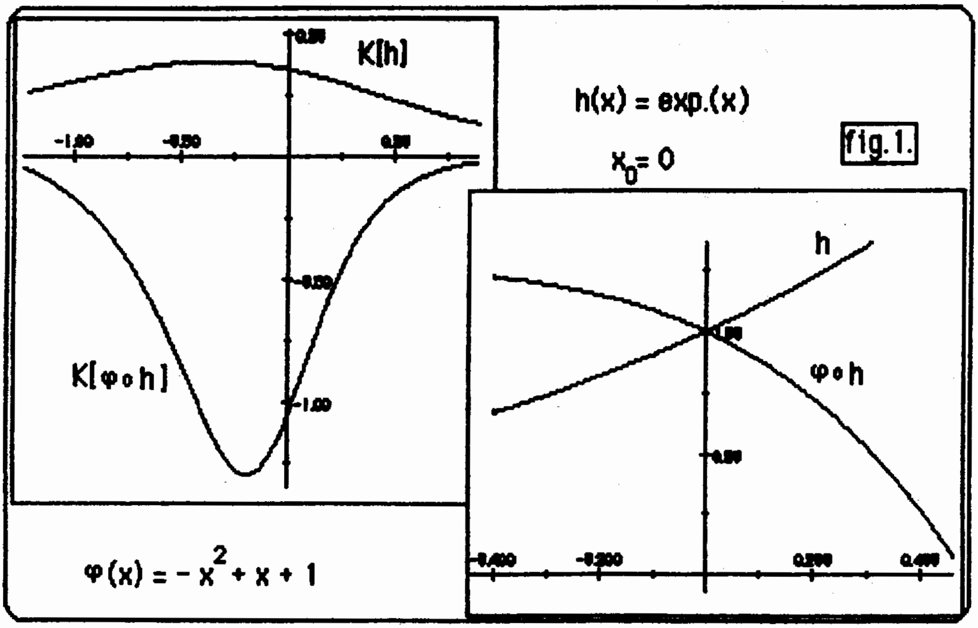


Figure 1:

at each point, note that if $\varphi$ is a concave function in $C^{(2)}(U), h(x_0) \in U$, with $\varphi'(h(x_0)) = 1$, it is clear that $\varphi \circ h$ has a smaller curvature than $h$, fig.1., since that

$$K(\varphi \circ h)(x_0) = K(h)(x_0) + \frac{\varphi''(h(x_0))h'(x_0)^2}{[1 + h'(x_0)^2]^{3/2}}$$

Hence, by applying a concave operator to a convex function we obtain a function with a smaller curvature than the original convex function.

Taking this account, if we consider the logarithmic function and the convex function $T[h]$, with $T[h](x) = h(x) - h(x_0) + 1$, we are in the previous conditions. Now, if we apply the logarithmic operator to $T[h]$ sucessively, until we obtain a concave function, we can define an index of the convexity measure of a function at each point considering the number of times that we need to apply the logarithmic operator to get a concave

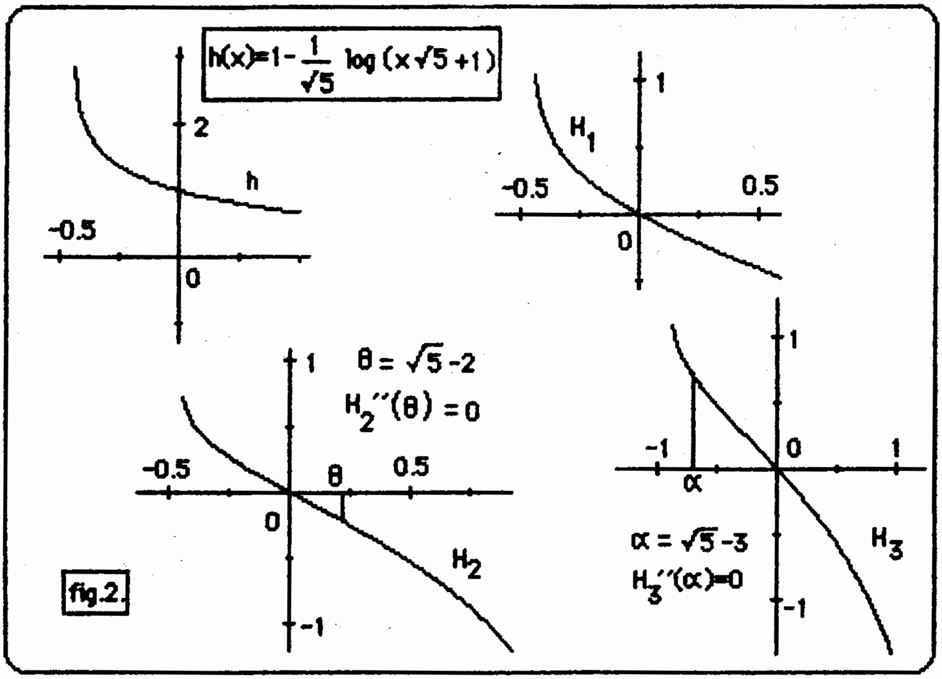


Figure 2:

function. So, if we define $H_n(x) = \log G_{n-1}(x)$ with $G_0(x) = T[f](x)$ and $G_n(x) = T[H_n](x)$ for $n \geq 1$, the sequence $\{H_n(x)\}$ will characterize the log-degree of convexity of $h$, since $K(h)(x_0) > K(H_1)(x_0) > \ldots > K(H_n)(x_0) \geq 0 > K(H_{n+1})(x_0) > \ldots$ (fig. 2.)

On the other hand, it is easy to prove by induction that $H_n''(x_0) = h''(x_0) - nh'(x_0)^2$ for all $n$, and therefore it follows that $H_n$ is convex at $x_0$ if and only if $h''(x_0)[h'(x_0)]^{-2} \geq n$. When $x_0$ is a minimum of $f$, it follows that $H_n$ is convex at $x_0$ for all $n$, and conversely. Then, the log-degree of convexity of $h$ at $x_0$ is defined to be the positive real number given by

$$U[h](x_0) = h''(x_0)[h'(x_0)]^{-2} \tag{1}$$

If $x_0$ is a minimum of $h$, we set $U[h](x_0) = +\infty$.

Note that there exists a similar behaviour between the curvature and the log-degree of convexity of a function, except in a neighbourhood of the critical points; a situation that it is not present in our conditions.

On the other hand, Roberts [5] points out some of the "good properties" that a measure of the convexity should have. It is easy to prove that $U[h](x)$ satisfies some of these.

(i) $U[h](x_0) \geq 0$ and $U[h](x) = 0$ in $V$ if and only if $h$ is affine.

(ii) In our conditions, i.e., $h_1$ and $h_2$ increasing functions, it is verified that $U[h_1 + h_2](x_0) \leq U[h_1](x_0) + U[h_2](x_0)$.

Finally, we are going to illustrate that $U[h]$ is a good index of convexity measure with some examples about power functions. If we consider the functions: $x^2, x^3$ and $x^4$ in a neighbourhood of $x_0 = 2$, where these functions fulfil our conditions, it is easy to observe (fig. 3.) that their log-degree of convexity and the curvature have an analogous behaviour.

## 3. Whittaker's method and convexity

Whittaker's method, [4], consists in applying the iterative process

$$(2) \qquad x_n = F(x_{n-1}) \quad \text{with} \quad F(x) = x - \lambda f(x)$$

When the method is convergent, i.e. $0 < \lambda \leq [f'(b)]^{-1}$, we obtain a decreasing sequence $\{x_n\}$, such that $\lim_n x_n = s$, and this convergence is linear.

The following results point out the influence of convexity in the convergence of $\{x_n\}$.

**Theorem 1.** *Assume $f$ and $g$ as in the Introduction, $U[g](x) < U[f](x)$ in $[a,b]$ and let $h(x) = kg(x)$, where $k = \max\{1, f'(b)/g'(b)\}$. Then, the sequence $\{y_n\}$ defined by $y_n = y_{n-1} - \lambda h(y_{n-1})$, with $y_0 = x_0$ and $\lambda \leq \min\{[g'(b)]^{-1}, [f'(b)]^{-1}\}$, is converges to $s$ and $y_n < x_n$ for all $n$.*

*Proof.* If we suppose that $f'(b) \leq g'(b)$, then the sequences $\{x_n\}$ defined in (2) and $\{y_n\}$ are two decreasing sequences to $s$ when $\lambda \leq [g'(b)]^{-1}$. We shall prove that $y_n < x_n$, by using the induction.

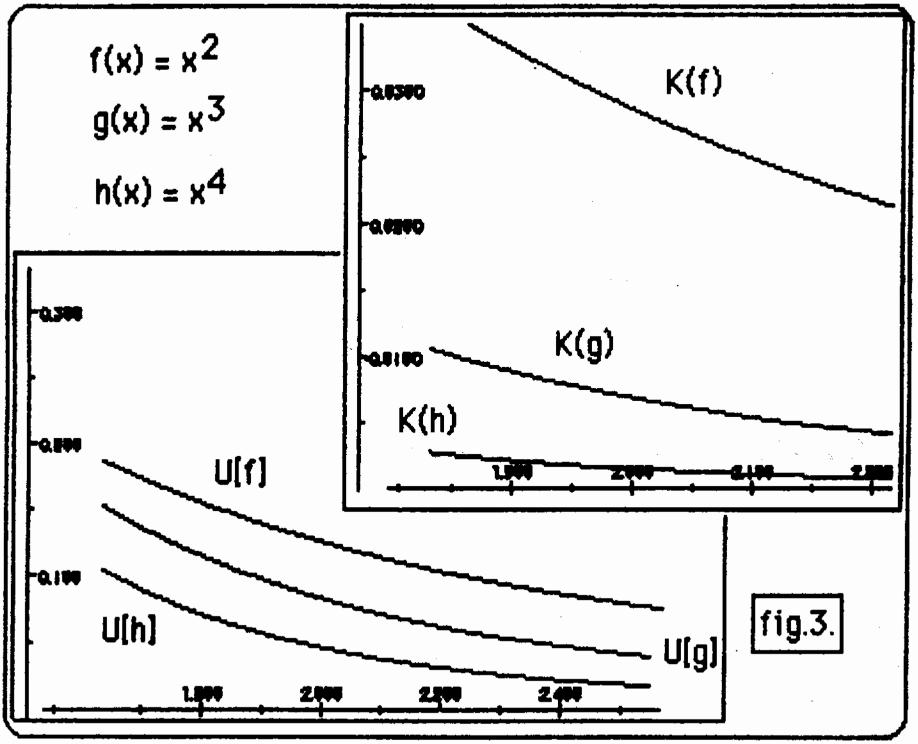


Figure 3:

Since $U[g](x) < U[f](x)$ in $[a,b]$, then $g'(x) > f'(x)$ in $(a,b)$. Moreover, it follows that $g(x) > f(x)$ in $(s,b)$, and therefore

$$x_1 - y_1 = \lambda[g(x_0) - f(x_0)] > 0$$

Now, assume $x_k > y_k$ for $k = 1,2,\ldots,n-1$. Since $F$ is an increasing function we have that $x_n - y_n \geq \lambda[g(y_{n-1}) - f(y_{n-1})]$. In a similar way to the case $n = 1$, we obtain that $x_n > y_n$.

If $g'(b) < f'(b)$, we consider $h(x) = f'(b)[g'(b)]^{-1}g(x)$. Then, $U[h](x) < U[f](x)$ in $[a,b]$ with $h'(b) = f'(b)$, and the result is proved by applying the above arguments to function $h$.

Since the best approximation in Whittaker's method is obtained when $\lambda = [f'(b)]^{-1}$, we are going to take an acceleration of Whittaker's method for this $\lambda$. By reasoning as in the Introduction, as $g'(b) \sim f'(s) \sim f'(x_{n-1})$, we get.

**Corollary 1.** *Let* $m \geq 3$ *with* $f'''(x) \geq 0$ *in* $[a,b]$*. Then, there exists* $n_0 \in N$ *such that the sequence*

$$y_n = x_{n-1} - \frac{f(x_{n-1})}{2f'(x_{n-1})}[2 - f(x_{n-1})U[f](x_{n-1})] \tag{3}$$

*is an acceleration of* $\{x_n\}$ *for* $n > n_0$.

*Proof.* Denote $L_f(x) = f(x) \cdot U[f](x)$ and let $F(x) = x - \frac{f(x)}{2f'(x)}[2 - L_f(x)]$. Since $F'(x) = \frac{L_f(x)}{2}[4 - 3L_f(x)] + \frac{f(x)^2}{2f'(x)^3}f'''(x)$, then $F$ is an increasing function in a neghbourhood of $s$, and so $\{y_n\}$ is a decreasing sequence to $s$.

Note that as

$$x_n - y_n = f(x_{n-1})[\frac{1}{f'(x_{n-1})}(1 - \frac{f(x_{n-1})U[f](x_{n-1})}{2}) - \frac{1}{f'(b)}],$$

then there exists $n_0 \in N$ such that $x_n > y_n$.

If $x_0$ is a good approximation to $s$, the previous acceleration is effective from the first iteration. This is the convex acceleration of Whittaker's method and it enables us to define a new iterative process, and to give a theorem of convergence.

**Theorem 2.** *Let $f$ be as before with $m \geq 3$. If $U[f](x) < 1/f(b)$ and $U[f'](x) \geq (-1)/f'(a)$ in $[a,b]$, then the iterative process*

$$(4) \qquad x_n = x_{n-1} - \frac{f(x_{n-1})}{2f'(x_{n-1})}[2 - f(x_{n-1})U[f](x_{n-1})]$$

*yields a decreasing sequence which converges to $s$.*

*Proof.* If $F$ is as in the Corollary, then

$$F'(x) = \frac{L_f(x)}{2}[(4 - L_f(x)) + L_f(x)(f'(x)U[f'](x) - 2)]$$

Since $L_f(x) < 1$ in $(s,b)$, then $F'(x) \geq 0$ if $f'(x)U[f'](x) \geq -1$, and therefore $F$ is an increasing function in $(s,b)$.

As $x_1 - s = F(x_0) - F(s) = F'(\xi_0)(x_0 - s)$ for $\xi_0 \in (s, x_0)$, it follows that $x_1 \geq s$. Repeating the reasoning we obtain that $x_n \geq s$ for all $n$. Moreover, it is obvious that $\{x_n\}$ is a decreasing sequence and therefore there exists $\lim_n x_n = u$, and since $f(u)U[f](u) < 1$, one can conclude that $u = s$ taking the limits in (4).

If $m \geq 4$, it is easy to pove that $F(s) = s$, $F'(s) = 0$ and $F''(s) \neq 0$, from which it follows that the iteration (4) is of the second order.

Notice that we have used the condition $f'(x)U[f'](x) \geq -1$, which is weaker than $f'''(x) \geq 0$.

Below we are going to study the convex acceleration of this new iterative process. Let $f$, $g$ be as before, and defiene $C(x) = g(x)[f(x)]^{-1}$ if $x \neq s$ and $C(s) = g'(s)[f'(s)]^{-1}$. Denote $M = \max\{C(x)/x \in [a,b]\}$.

**Theorem 3.** *Let $x_n = F(x_{n-1})$ and $y_n = G(y_{n-1})$, where $F(x) = x - \frac{f(x)}{f'(x)}H(L_f(x))$, $G(x) = x - \frac{g(x)}{g'(x)}H(L_g(x))$ and $H(x) = 1 - \frac{x}{2}$. Consider $f$ and $g$ as in Theorem 2, with $U[f](x) > MU[g](x)$ and $x_0 = y_0 \in [a,b]$, Then $y_n < x_n$ for all $n$.*

*Proof.* As $L_f(x) > L_g(x)$ in $(s,b)$, then $f(x)/f'(x) < g(x)/g'(x)$. Hence, $F(x) > G(x)$ in $(s,b)$ and by using the induction, $x_n > y_n$ holds for all $n$.

Now, proceeding as in the Introduction, we obtain that the convex acceleration of $\{x_n\}$ in (4) is

$$(5)\, y_n = x_{n-1} - \frac{f(x_{n-1})}{4f'(x_{n-1})}[2 - L_f(x_{n-1}) + \frac{4 + 2L_f(x_{n-1})}{2 - L_f(x_{n-1})(2 - L_f(x_{n-1}))}]$$

This acceleration provides us with a new iterative process. It is known [2], that if we have an iterative process $x_n = F(x_{n-1})$ with $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} H(L_f(x_{n-1}))$ and $H(0) = 1, H'(0) = \frac{1}{2}$ and $|H''(0)| < +\infty$, it has a cubic convergence. Then, if $m \geq 4$, it is easy to prove that the iterative process given by the analogous to (5) is of the third order.

## 4. Practical remark

By applying the iterative process (4), it is necessary to compute $f$, $f'$ and $f''$ for each step, whereas in Whittaker's method it is only necessary to compute $f$ for each step and $f'$ in a fixed point (moreover of specific calculations of each process). But, the calculations required have not an negative practical influence since, for the solution of ordinary equations, Whittaker's method (linear convergence) needs a number of iterations bigger than the iterative process (4). Then, the time that the computer takes to get a good approximation to the solution is minor with this new iterative process. Besides, we obtain a very good approximation.

Repeating these previous reasonings for the iterative process obtained from (5), we get the same results, since this new process is of the third order.

Example: We are going to comparate these new iterative processes applied to the Wallis polynomial equation, fig. 4., with respect to the Whittaker method.

The computer time of each process is shorter than in the Whittaker method, fig.4, and it provides us a better approximation to the solution of the equation (the solution to 18 decimal places is $\gamma = 2.094551481542326591$, [4]). If we consider another tolerance between the iterations, we can obtain a better approximation to the solution with the Whittaker method, but the computer time increases.

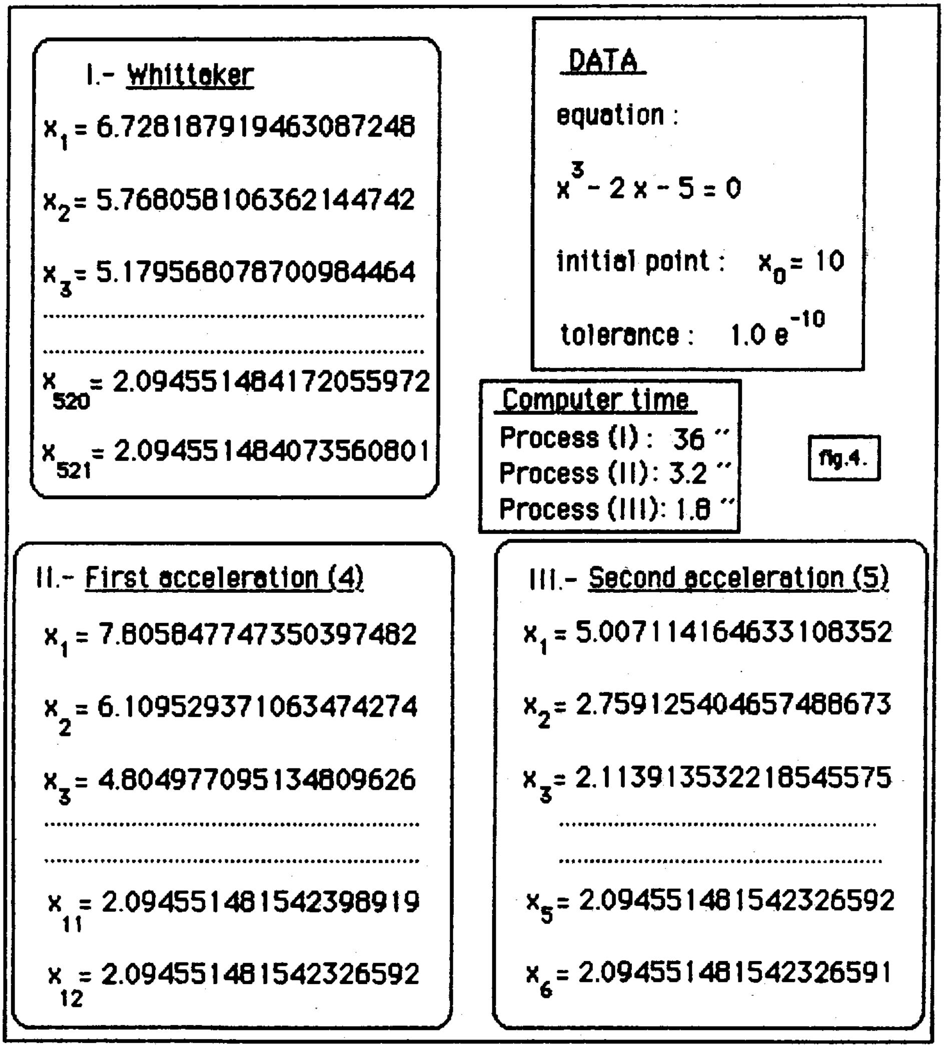
I.- Whittaker
x1 = 6.728187919463087248
x2 = 5.768058106362144742
x3 = 5.179568078700984464
x520 = 2.094551484172055972
x521 = 2.094551484073560801
DATA
equation :
x3 - 2 x - 5 = 0
initial point : x0 = 10
tolerance : 1.0 e-10
Computer time
Process (I) : 36 ''
Process (II): 3.2 ''
Process (III): 1.8 ''
fig.4.
II.- First acceleration (4)
x1 = 7.805847747350397482
x2 = 6.109529371063474274
x3 = 4.804977095134809626
x11 = 2.094551481542398919
x12 = 2.094551481542326592
III.- Second acceleration (5)
x1 = 5.007114164633108352
x2 = 2.759125404657488673
x3 = 2.113913532218545575
x5 = 2.094551481542326592
x6 = 2.094551481542326591

Figure 4:

## 5. General remarks

If function $f$ is decreasing, all the previous results turn out to be valid by changing slightly the used reasoning.

The condition $f(x_0) > 0$ does not affect the results obtained. If $f(x_0) < 0$, then the only change is that the sequence $\{x_n\}$ is increasing.

If function $f$ is concave, then we have to consider the corresponding exp-degree of concavity and, by applying the previously studied reasoning, we obtain analogous results.

## References

[1] Ehrmann, H., Konstruktion und durchduhrung von iterationsverfaharen hoherer ordnung, *Arch. Rational Mech. Anal.* 4 (1959) 65-88.

[2] Gander, W., On Halley's iteration method, *Amer. Math. Monthly Vol.* 92 No 2 (1985) 131-134

[3] O'Neill, B., Elementary differential geometry, *Academic Press, New York,* 1966.

[4] Ostrowski, A.M., Solution of equations and systems of equations, *Academic Press, New York,* 1973.

[5] Roberts, A.W. and Varberg, D.E., Convex functions, *Academic Press, New York,* 1973

[6] Traub, J.F., Iterative methods for solution of equations, *Prentice-Hall, Englewood Cliffs*, 1964.

**REZIME**

JEDNA UBRZANA PROCEDURA ZA WHITTAKEROV METOD POMOĆU KONVEKSNOSTI

U radu se posmatra uticaj konveksnosti realne funkcije $f$ pri primeni Whittakerove metode na rešavanje jednačine $f(x) = 0$ i dobijen je metod za ubrzanje ovog iterativnog procesa. Takodje su dobijeni neki novi iterativni procesi.

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