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DIHEDRAL n-QUASIGROUPS

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Abstract

An n-quasigroup (Q, f) is called dihedral iff $f(x_1, ..., x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma(1)}, ..., x_{\sigma(n)}) = x_{\sigma(n+1)}$ for every permutation $\sigma \in D_{n+1}$, where D_{n+1} is the dihedral subgroup of the symmetric group S_{n+1} of degree $n+1$. Dihedral n-quasigroups (D-n-quasigroups) represent a generalization of totally symmetric binary quasigroups. Several equivalent definitions and some examples of D-n-quasigroups are given. It is proved that some retracts of D-n-quasigroups are also D-n-quasigroups.Autotopisms and regular permutations of D-n-quasigroups are cosidered and some of their properties determined.

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1. Definitions and Notations

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_m, x_{m+1}, \ldots, x_n$ we shall denote by x_m^n or $\{x_i\}_{i=m}^n$. If $m > n$, then x_m^n will be considered empty. The sequence $x, x, \ldots, x$ (m times) will be denoted by $\overset{m}{x}$. If $m \leq 0$, then $\overset{m}{x}$ will be considered empty.

An n-ary groupoid (n-groupoid) (Q, f) is called an n-quasigroup iff the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in \{1, \ldots, n\} = N_n$.

An n-quasigroup (Q, f) is isotopic to an n-quasigroup (Q, g) iff there exists a sequence $T = (\alpha_1^{n+1})$ of permutations of Q such that the following identity

$$g(x_1^n) = \alpha_{n+1}^{-1} f(\{\alpha_i x_i\}_{i=1}^n)$$

holds. T is called an isotopism, g is an isotope of f and by $f^T = g$ we denote that f is isotopic to g by T. T^{-1} is defined by $T^{-1} = (\{\alpha_i^{-1}\}_{i=1}^{n+1})$. If T is an isotopism of (Q, f) to itself, that is, $f^T = f$, then T is called an autotopism of f. The set of all autotopisms of f under the composition of autotopisms is a group which we denote by $A(f)$.

By S_n we denote the symmetric group of degree n, by A_n its alternating subgroup, and by D_n its dihedral subgroup.

If (Q, f) is an n-quasigroup and $\sigma \in S_{n+1}$, then the n-quasigroup f^σ defined by

$$f^\sigma(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)} \iff f(x_1^n) = x_{n+1}$$

is called a σ-parastrophe (or simply parastrophe) of f. If $f = f^\sigma$, then σ is called an autoparastrophism of f. The set of all autoparastrophisms of f is a subgroup of S_{n+1} which is denoted by $\Pi(f)$.

An n-quasigroup (Q, f) is called

a) totally symmetric (TS) if $f = f^\sigma$ for all $\sigma \in S_{n+1}$,

b) alternating symmetric (AS) iff $f = f^\sigma$ for all $\sigma \in A_{n+1}$ ([8]),

c) cyclic iff $f = f^\sigma$ for all $\sigma \in C$, where C is a subgroup of S_{n+1} generated by the cycle $(12 \ldots n+1)$ ([7],[10]).

If Q is a set, by ϵ we denote the identity mapping of Q.

2. Dihedral n-quasigroups

Definition 1. *An n-groupoid (Q, f) is called dihedral iff for every permutation $\sigma \in D_{n+1}$ and every $x_1^{n+1} \in Q$*

$$f(x_1^n) = x_{n+1} \iff f(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)}.$$

It is not difficult to see that from the preceding definition it follows that every dihedral n-groupoid is necessarily an n-quasigroup, which we call a dihedral n-quasigroup (D-n-quasigroup).

Since every D-n-groupoid is a D-n-quasigroup, Definition 1 can be given in another form.

Definition 1'. *An n-quasigroup (Q, f) such that $f = f^{\sigma}$ for every $\sigma \in D_{n+1}$ is a D-n-quasigroup.*

In the symmetric group S_{n+1} the subgroup D_{n+1} is generated by permutations $\phi = (12\ldots n{+}1), \psi = \prod_{2 \le i < n+3-i}(i \quad n{+}3{-}i)$, which implies the following theorem.

In the sequel ϕ and ψ will always denote these two permutations.

Theorem 1. *An n-quasigroup (Q, f) is a D-n-quasigroup iff $f = f^{\phi} = f^{\psi}$.*

D-n-quasigroups can also be defined as n-quasigroups satisfying a system of identities. Since every equality of type $f = f^{\sigma}$, where $\sigma \in S_{n+1}$, is equivalent to an identity ([5]), from Theorem 1 we get the next theorem.

Theorem 2. *An n-groupoid (Q, f) is a D-n-quasigroup iff for all $x_1^{n+1} \in Q$*

$$\begin{cases} f(x_2^n, f(x_1^n)) = x_1, \\ f(x_1, f(x_1^n), x_n, x_{n-1}, \ldots, x_3) = x_2. \end{cases}$$

Since D-n-quasigroups can be defined by a system of identities, the class of all D-n-quasigroups is a variety, which implies that this class is closed under the formation of direct products, subalgebras and quotient algebras.

For $n = 2$ a D-n-quasigroup (Q, f) is a binary quasigroup $(Q, \cdot)$ satisfying the identities

$$y(xy) = x, \quad x(xy) = y$$

which implies that $(Q, \cdot)$ is a TS binary quasigroup. Hence D-n-quasigroups represent a generalization of TS binary quasigroups, different from other generalizations such as TS-n-quasigroup or AS-n-quasigroups.

For $n = 3$ a class of generalized idempotent D-3-quasigroups (see [9]) is equivalent to a new class of quadruple systems called dihedral quadruple systems, which lies between Steiner and Mendelsohn quadruple systems.

Since $C_{n+1} \subset D_{n+1} \subset S_{n+1}$, every TS-n-quasigroup is dihedral and every D-n-quasigroup is cyclic. If we denote the class of all TS-n-quasigroup by $\mathcal{A}(TS)$, the class of all D-n-quasigroups by $\mathcal{A}(D)$ and the class of all cyclic n-quasigroups by $\mathcal{A}(C)$, it follows

$$\mathcal{A}(TS) \subset \mathcal{A}(D) \subset \mathcal{A}(C).$$

That the class $\mathcal{A}(D)$ is different from both $\mathcal{A}(TS)$ and $\mathcal{A}(C)$ follows from [2], where it was proved that for every subgroup $G \subset S_{n+1}$ and every $m > n$, $p \geq 2$, there exists an n-quasigroup (Q, f) of order mp such that $\Pi(f) = G$.

If $n \equiv 0 \pmod 4$, then $D_{n+1} \subset A_{n+1}$, hence $\mathcal{A}(AS) \subset \mathcal{A}(D)$, where $\mathcal{A}(AS)$ denotes the class of all AS-n-quasigroups. So, in this case we have

$$\mathcal{A}(TS) \subset \mathcal{A}(AS) \subset \mathcal{A}(D) \subset \mathcal{A}(C).$$

For other values of n D_{n+1} is not contained in A_{n+1}.

Now we shall give some examples of D-n-quasigroups.

1. If $(G, +)$ is a commutative group, θ an automorphism of $(G, +)$ such that $\theta^2 = \epsilon$, b an element of G such that $\theta b = b$, n odd, then by

$$f(x_1^n) = \theta x_1 - x_2 + \theta x_3 - x_4 + \ldots + \theta x_n + b$$

a D-n-quasigroup (G, f) is defined.

2. Let the Euclidean plane be given and let A, B and C be arbitrary points (not necessarily distinct) in the plane. If D is the fourth vertex of the parallelogram $ABCD$ determined by vertices A, B and C (which can be degenerated if some of the points A, B and C coincide), then we define a ternary mapping f by

$$f(A, B, C) = D.$$

If S denotes the set of all points in the plane, then $f : S^3 \to S$ and it is not difficult to see that (S, f) is a D-3-quasigroup of infinite order which is not totally symmetric.

In the preceding example an infinite uncountable D-3-quasigroup was constructed. An analogous construction on the set of all points in a plane with integer coordinates gives an example of an infinite countable D-3-quasigroup.

3. As we have mentioned before, in [2] a method which enables a construction of D-n-quasigroups of order mp for every $m > n$, $p \geq 2$, such that $\Pi(f) = D_{n+1}$, was given.

3. Retracts of D-n-quasigroups

We shall prove that every D-n-quasigroup, where n is odd, defines a family of D-($\frac{n+1}{2}$)-quasigroups.

Theorem 3. *If (Q,f) is an D-n-quasigroup, where $n = 2k+1$, $k \in \{2,3,\ldots\}$, and a is an arbitrary element from Q, then by*

$$g(x_1^k) = x_{k+1} \iff f(a,x_1,a,x_2,\ldots,a) = x_{k+1}$$

a D-k-quasigroup (Q,g) is defined.

Proof. g is a k-quasigroup since it is a retract of an n-quasigroup. From the cyclicity of f it follows

$$f(a,x_1,a,x_2,\ldots,a) = x_{k+1} \iff f(a,x_{k+1},a,x_1,\ldots,a) = x_k,$$

and

$$g(x_1^k) = x_{k+1} \iff g(x_{k+1},x_1^{k-1}) = x_k,$$

which means that g is also cyclic.

Since f is dihedral $f = f^{\psi}$, where $\psi = \prod_{2\le i<n+3-i}(i \quad n+3-i)$, hence

$$f(a,x_1,a,x_2,\ldots,a) = x_{k+1} \iff f(a,x_{k+1},a,x_k,\ldots,a) = x_1$$

and

$$g(x_1^k) = x_{k+1} \iff g(x_{k+1},x_k,\ldots,x_2) = x_1.$$

Since g is cyclic, we get

$$g(x_1^k) = x_{k+1} \iff g(x_1,x_{k+1},x_k,\ldots,x_3) = x_2,$$

We have proved that $g = g^{\psi'}$, where $\psi' = \prod_{2\le i<k+3-i}(i \quad k+3-i)$, so by Theorem 1 g is dihedral.

4. Autotopisms of D-n-quasigroups

Lemma 1. *If $T = (\alpha_1^{n+1})$ is an autotopism of a D-n-quasigroup (Q,f), then for every $\sigma \in D_{n+1}$, $T^{\sigma} = (\{\alpha_{\sigma(i)}\}_{i=1}^{n+1})$ is also an autotopism of f.*

Proof. From $f^T = f^\sigma = f$, it follows that $f = (f^T)^\sigma = (f^\sigma)^{T^\sigma} = f^{T^\sigma}$. Hence T^σ is an autotopism of f.

In the theory of binary, as well as n-ary quasigroups, there exists a close relation between nuclei and groups of regular permutations. There exist several different generalizations of nuclei and regular permutations to n-ary case, here we shall consider regular permutations as defined in [6],[1],[3],[4].

Let (Q, f) be an n-quasigroup, $i \in N_n$. A permutation α of Q is said to be i-inverse regular for f iff $(\overset{i-1}{\epsilon}, \alpha, \overset{n-i}{\epsilon}, \alpha^{-1}) \in A(f)$. A permutation which is i-inverse regular for every $i \in N_n$ is called inverse regular for f. The set of all inverse regular permutations for f will be denoted by $V(f)$.

Theorem 4. *If (Q, f) is a D-n-quasigroup, n even, $T = (\overset{i-1}{\epsilon}, \alpha, \overset{j-i-1}{\epsilon}, \beta, \overset{n-j+1}{\epsilon}) \in A(f)$ for some $i, j \in N_n$, then $\beta = \alpha^{-1}$.*

Proof. a) $j - i$ odd.

Then there exists $t \in N_n$ such that T^{ϕ^t} or $T^{\phi^{-t}}$ has the following form: $S = (\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \beta, \overset{n-m+1}{\epsilon})$, where $k + m = n + 3$. The last condition implies that $(k \quad m)$ is one of the transpositions from the product $\psi = \prod_{2 \leq i < n+3-i} (i \quad n+3-i)$. By Lemma 1 $S \in A(f)$, hence the following identity is valid

$$f(x_1^n) = f(x_1^{k-1}, \alpha x_k, x_{k+1}^{m-1}, \beta x_m, x_{m+1}^n). \tag{1}$$

Putting in (1) $x_1 = \ldots = x_n = x$, and since $f = f^\psi$, we get

$$f(\overset{k-1}{x}, \alpha^{-1}x, \overset{n-k}{x}) = f(\overset{m-1}{x}, \beta x, \overset{n-m}{x}) = f(\overset{k-1}{x}, \beta x, \overset{n-k}{x}).$$

Hence $\alpha^{-1}x = \beta x$, that is, $\beta = \alpha^{-1}$.

b) $j - i$ even.

In this case there exists $t_1 \in N_n$ such that $T^{\phi^{t_1}}$ or $T^{\phi^{-t_1}}$ is of the form $S_1 = (\overset{k-1}{\epsilon}, \beta, \overset{m-k-1}{\epsilon}, \alpha, \overset{n-m+1}{\epsilon})$, where $k + m = n + 3$, and the rest of the proof is analogous to the preceding case.

Theorem 5. *Let (Q, f) be a D-n-quasigroup, n odd. If $T = (\overset{i-1}{\epsilon}, \alpha, \overset{j-i-1}{\epsilon}, \beta, \overset{n-j+1}{\epsilon}) \in A(f)$ for some $i, j \in N_n$ such that $j - i$ is even, then $\beta = \alpha^{-1}$.*

Proof. Then there exists $t \in N_n$ such that T^{ϕ^t} or $T^{\phi^{-t}}$ has the form $S = (\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \beta, \overset{n-m+1}{\epsilon})$ where $k + m = n + 3$. The rest of the proof is analogous to the proof of Theorem 4.

Theorem 6. *If (Q, f) is a D-n-quasigroup and for some $i \in N_n$ $(\overset{i-1}{\epsilon}, \alpha, \alpha^{-1}, \overset{n-i}{\epsilon}) \in A(f)$, then $(\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \alpha^{-1}, \overset{n-m+1}{\epsilon}) \in A(f)$ for every $k, m \in N_{n+1}$, $k < m$.*

Proof. If we denote $T_i = (\overset{i-1}{\epsilon}, \alpha, \alpha^{-1}, \overset{n-i}{\epsilon})$, then $T_i^{\phi^{-p}} = T_{i+p} = (\overset{i+p-1}{\epsilon}, \alpha, \alpha^{-1}, \overset{n-i-p}{\epsilon}) \in A(f)$ for all $p = 1 - i, 2 - i, \ldots, n - i$. Hence $S_k = T_n T_{n-1} \ldots T_k = (\overset{k-1}{\epsilon}, \alpha, \overset{n-k}{\epsilon}, \alpha^{-1}) \in A(f)$ for every $k \in N_n$, and $S_k S_m^{-1} = (\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \alpha^{-1}, \overset{n-m+1}{\epsilon}) \in A(f)$ for every $k, m \in N_{n+1}$, $k < m$.

From Theorems 4 and 6 we get the following corollary.

Corollary 1. *If (Q, f) is a D-n-quasigroup, n even, and for some $i \in N_n$ $(\overset{i-1}{\epsilon}, \alpha, \beta, \overset{n-i}{\epsilon}) \in A(f)$, then $(\overset{k-1}{\epsilon}, \alpha, \overset{m-k-1}{\epsilon}, \alpha^{-1}, \overset{n-m+1}{\epsilon}) \in A(f)$ for every $k, m \in N_{n+1}$, $k < m$.*

By $A_i(f)$ we shall denote the set of all i-th components of all autotopisms of the n-quasigroup (Q, f). $A_i(f)$ is obviously a group under the composition of permutations.

Theorem 7. *If (Q, f) is a D-n-quasigroup, then $A_i(f) = A_j(f)$ for all $i, j \in N_{n+1}$.*

Proof. Let $i, j \in N_{n+1}$ and let $T = (\alpha_1^{n+1})$ be an arbitrary autotopism of a D-n-quasigroup (Q, f). Then by Lemma 1 $T^{\phi^{j-i}}$ is also an autotopism of f such that its j-th component is α_i. So, $A_i(f) \subset A_j(f)$, and analogously it follows $A_i(f) \supset A_j(f)$, hence $A_i(f) = A_j(f)$.

The group of all components of all autotopisms of a D-n-quasigroup (Q, f) we shall denote by $A_0(f)$.

Theorem 8. *If (Q, f) is a D-n-quasigroup, n even, then $V(f)$ is a commutative group.*

Proof. If $\alpha,\beta \in V(f)$, then $T = (\alpha,\overset{n-1}{\epsilon},\alpha^{-1}) \in A(f)$ and $S = (\beta,\overset{n-1}{\epsilon},\beta^{-1}) \in A(f)$. Then $TS = (\alpha\beta,\overset{n-1}{\epsilon},\alpha^{-1}\beta^{-1}) \in A(f)$ and $(TS)^{\phi^{-1}} = (\alpha^{-1}\beta^{-1},\alpha\beta,\overset{n-1}{\epsilon}) \in A(f)$ hence by Theorem 4 $\alpha^{-1}\beta^{-1} = (\alpha\beta)^{-1}$, that is, $\alpha\beta = \beta\alpha$. Since $((TS)^{\phi^{-1}})^{-1} = (\alpha\beta,(\alpha\beta)^{-1},\epsilon) \in A(f)$, by Theorem 6 it follows that $\alpha\beta \in V(f)$. From $T^{\phi^{-1}} = (\alpha^{-1},\alpha,\overset{n-1}{\epsilon}) \in A(f)$ by Theorem 6 $\alpha^{-1} \in V(f)$.

An autotopism of an n-quasigroup (Q,f) is called inverse regular if all its components are inverse regular for f. The set of all inverse regular autotopisms of a D-n-quasigroup (Q,f), where n is even, is obviously a commutative group which will be denoted by $A'(f)$.

Theorem 9. *If (Q,f) is a D-n-quasigroup, n even, then $V(f)$ and $A'(f)$ are normal subgroups in groups $A_0(f)$ and $A(f)$ respectively.*

Proof. If $\alpha \in V(f)$, then $T = (\alpha,\overset{n-1}{\epsilon},\alpha^{-1}) \in A(f)$ and if $\beta \in A_0(f)$, then there exists an autotopism of f of the form $S = (\alpha_1^n,\beta)$. Then $R = S^{-1}TS = (\alpha_1^{-1}\alpha\alpha_1,\overset{n-1}{\epsilon},\beta^{-1}\alpha^{-1}\beta) \in A(f)$ and $R^{\phi^{-1}} = (\beta^{-1},\alpha\beta,\alpha_1^{-1}\alpha\alpha_1,\overset{n-1}{\epsilon})$ which by Theorems 4 and 6, implies that $\beta^{-1}\alpha\beta \in V(f)$, hence $V(f)$ is a normal subgroup of $A_0(f)$.

From the preceding it follows that all components of an autotopism of the form $S^{-1}TS$, where $S \in A(f), T \in A'(f)$, are inverse regular permutations, hence $A'(f)$ is a normal subgroup of $A(f)$.

Theorem 10. *If $(\alpha,\beta,\gamma,\overset{n-2}{\epsilon})$ is an autotopism of a D-n-quasigroup (Q,f), n even, then $\alpha\gamma^{-1}$ is inverse regular for f.*

Proof. From $T = (\alpha,\beta,\gamma,\overset{n-2}{\epsilon}) \in A(f)$, it follows that $S = T^{\psi\phi^{-2}} = (\gamma,\beta,\alpha,\overset{n-2}{\epsilon})$. Then $TS^{-1} = (\alpha\gamma^{-1},\epsilon,\gamma\alpha^{-1},\overset{n-2}{\epsilon}) \in A(f)$, and $(TS^{-1})^{\phi^{-2}} = (\overset{2}{\epsilon},\alpha\gamma^{-1},\epsilon,\gamma\alpha^{-1},\overset{n-4}{\epsilon}) \in A(f)$. Continuing this, we obtain that $R = (\alpha\gamma^{-1},\overset{n-1}{\epsilon},\gamma\alpha^{-1}) \in A(f)$ and $(R^{\phi^{-1}})^{-1} = (\alpha\gamma^{-1},\gamma\alpha^{-1},\overset{n-1}{\epsilon}) \in A(f)$, which by Theorem 6 implies that $\alpha\gamma^{-1}$ is inverse regular for f.

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REZIME

DIEDARSKE n-KVAZIGRUPE

n-kvazigrupa (Q, f) se naziva diedarska ako je $f(x_1, ..., x_n) = x_{n+1} \Leftrightarrow f(x_{\sigma(1)}, ..., x_{\sigma(n)}) = x_{\sigma(n+1)}$ za svaku permutaciju $\sigma \in D_{n+1}$, gde je D_{n+1} diedarska podgrupa simetrične grupe S_{n+1} stepena $n + 1$. Diedarske n-kvazigrupe (D-n-kvazigrupe) predstavljaju generalizaciju totalno simetričnih binarnih kvazigrupa. Navedene su neke ekvivalentne definicije i dati primeri D-n-kvazigrupa. Dokazano je da su neki retrakti D-n-kvazigrupa takodje n-kvazigrupe. Razmatrane su autotopije i regularne permutacije D-n-kvazigrupa i odredjene neke njihove osobine.

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