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ON THE k-TH PRIME FACTOR OF AN INTEGER

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Abstract

Let $P_k(n)$ be the k-th largest prime factor of an integer $n \geq 1$ if n has at least k prime factors, and let $P_k(n)$ be zero otherwise. An asymptotic formula for the sum $\sum_{n\leq x} P_k(n)$ is obtained.

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Let $P_k(n)$ denote the k-th largest prime factor of an integer $n \geq 1$ if $\Omega(n) \geq k$, and let otherwise $P_k(n) = 0$, where $\Omega(n)$ is the number of all prime factors of n. In other words $P_k(1) = 0$, and if $P(n) = P_1(n)$ is the largest prime factor of $n \geq 2$, then $P_2(n) = P(n/P(n)), \ldots, P_k(n) = P(n/P_{k-1}(n))$ if $\Omega(n) \geq k$. In [1] Alladi and Erdös (see (1.21), p. 284) showed that, for $k \geq 1$ a fixed integer,

$$\sum_{n\leq x} P_k(n) = A_k \frac{x^{1+1/k}}{\log^k x} + O\left(\frac{x^{1+1/k}\log\log x}{\log^{k+1} x}\right). \tag{1}$$

Their proof was elementary, but quite complicated. Recently R. Balasubramanian [3] found a simple proof of (1) that works even if k is not fixed, but lies in a suitable interval depending on x. In addition, he evaluated

$A_k = k^{2k}\zeta(1+1/k)/(k+1)!$. In [4] J.-M. De Koninck and I proved, for any fixed integer $N \geq 1$,

$$\sum_{n\leq x} P_1(n) = A_{1,1}\frac{x^2}{\log x} + \ldots + A_{1,N}\frac{x^2}{\log^N x} + O(\frac{x^2}{\log^{N+1} x}), \tag{2}$$

where $A_{1,1} = \pi^2/12, \ldots, A_{1,N}$ are certain absolute constants which may be explicitly evaluated. The aim of this note is to generalize (2) as to include a sharper version of (1). The result is the following

Theorem. *Let $k \geq 1$ be a fixed integer, and $N \geq 0$ an arbitrary, but fixed integer. Then there exist constants $A_{k,1} > 0, \ldots, A_{k,N+1}$ which may be explicitly evaluated such that*

$$\sum_{n\leq x} P_k(n) = A_{k,1}\frac{x^{1+1/k}}{\log^k x} + \ldots + A_{k,N+1}\frac{x^{1+1/k}}{\log^{k+N} x} + O(\frac{x^{1+1/k}}{\log^{k+N+1} x}). \tag{3}$$

The proof of the asymptotic formula (3) will be also elementary. We shall make use of the formulas

$$\sum_{p\leq x}\frac{p^\alpha}{\log^\beta p} = c_1\frac{x^{\alpha+1}}{\log^{\beta+1} x} + \cdots + c_N\frac{x^{\alpha+1}}{\log^{\beta+N} x} + O(\frac{x^{\alpha+1}}{\log^{\beta+N+1} x}), \tag{4}$$

$$(\alpha > 0)$$

$$\sum_{y_1<p\leq y_2}\frac{1}{p^\alpha(\log\frac{x}{p})^\beta} = (\frac{d_1}{t^{\alpha-1}\log t(\log\frac{x}{t})^\beta} + \cdots \tag{5}$$

$$+\frac{d_N}{t^{\alpha-1}\log t(\log\frac{x}{t})^{\beta+N-1}})\Big|_{y_1}^{y_2} + O(\frac{1}{y_1^{\alpha-1}\log y_1(\log\frac{x}{y_1})^{\beta+N}}),$$

and

$$\sum_{n\leq x^\gamma}\frac{1}{n^\alpha(\log\frac{x}{n})^\beta} = \frac{e_1}{\log^\beta x} + \cdots + \frac{e_N}{\log^{\beta+N-1} x} + O(\frac{1}{\log^{\beta+N} x}), \tag{6}$$

where $0 < \gamma < 1$, and $\alpha > 1$ in (5) and (6). Here and in the sequel p (with or without indices) stands for primes, the constants $c_j = c_j(\alpha,\beta), d_j = d_j(\alpha,\beta), e_j = e_j(\alpha,\beta) (j = 1,\ldots,N)$ may be explicitly evaluated, $N \geq 1$ is an arbitrary fixed integer, and in (5) $y_1 < y_2 < x, y_1 \to \infty$. Moreover we shall use the standard notation $f = O(g)$ and $f << g$, which both mean that $|f| \leq Cg$ for some absolute $C > 0$. The proofs of the above formulas follow without difficulty from the prime number theorem and elementary analysis. For example, we have

$$\sum_{y_1<p\le y_2}\frac{1}{p^\alpha(\log\frac{x}{p})^\beta}=\int_{y_1+0}^{y_2-0}\frac{d\pi(t)}{t^\alpha(\log\frac{x}{t})^\beta}=\int_{y_1}^{y_2}\frac{dt}{t^\alpha(\log\frac{x}{t})^\beta\log t}$$

$$+O(y_1^{1-\alpha}(\log\frac{x}{y_1})^{-\beta-N}(\log y_1)^{-1})$$

by the prime number theorem in the standard form

$$\pi(x)=\sum_{p\le x}1=\int_2^x\frac{dt}{\log t}+O(xe^{-C\sqrt{\log x}})\qquad(C>0),$$

and successive integrations by parts give (5).

We begin the proof of (3) by noting that

$$\sum_{n\le x}P_k(n)=\sum_{mp_1p_2\dots p_k\le x,P(m)\le p_1,p_1\le p_2\le\dots\le p_k}p_1 \tag{7}$$

$$=\sum_{p_1p_2\dots p_k\le x,p_1\le p_2\le\dots\le p_k}p_1\psi(\frac{x}{p_1p_2\dots p_k},p_1),$$

where as usual $\psi(x,y)$ denotes the number of $n\le x$ all of whose prime factors are $\le y$. It will be shown now how the first equality in (7) may be used in a simple way to obtain lower and upper bounds for the sum of $P_k(n)$. Namely, let in (7) $m=1,x^{1/k}2^{-k}\le p_1\le x^{1/k}2^{1-k},\dots,\frac{1}{2}x^{1/k}\le p_k\le x^{1/k}$. By the prime number theorem there are $>>x^{1/k}/\log x$ choices for each $p_j(j=1,\dots,k)$. Therefore

$$\sum_{n\le x}P_k(n)>>x^{1/k}(x^{1/k}/\log x)^k=x^{1+1/k}\log^{-k}x, \tag{8}$$

which represents a lower bound of the right order of magnitude. On the other hand, we have

$$x\ge mp_1p_2\dots p_k\ge mp_1^k,$$

hence $p_1\le(x/m)^{1/k}$, which gives

$$\sum_{n\le x}P_k(n)\le x^{1/k}\sum_{m\le x}m^{-1/k}\sum_{p_1p_2\dots p_k\le x/m}1 \tag{9}$$

$$<<x^{1+1/k}\sum_{m\le x}m^{-1-1/k}\frac{(\log\log x)^{k-1}}{\log x}$$

$$\leq \zeta(1+1/k)x^{1+1/k}(\log\log x)^{k-1}\log x$$
$$<< x^{1+1/k}(\log\log x)^{k-1}\log x.$$

Here we used the classical elementary estimate

$$\sum_{n\leq x,\omega(n)=k} 1 < \frac{Ax(\log\log x+B)^{k-1}}{\log x(k-1)!}$$

$$(A,B>0; k\geq 1, x>2)$$

of Hardy and Ramanujan (see [7], p. 265), where A, B are suitable absolute constants and $\omega(n)$ is the number of distinct prime factors of n. Thus the upper bound in (9) is only by a factor of $(\log\log x)^{k-1}$ smaller than the true order of the summatory function of $P_k(n)$.

The main obstacle in evaluating the last sum in (7) is the presence of the ψ-function. However, noting that $\psi(x,y) = [x]$ if $y \geq x([x]$ is the integer part of $x)$ and using the trivial $\psi(x,y) \leq x$ for $y \leq x$, this difficulty may be overcome, and we may obtain essentially (7) without the condition $P_1(m) \leq p_1$. In this way the problem will be reduced to a rather technical one which involves a k-fold summation over prime s. We have

(10)
$$\sum_{p_1p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k} p_1\psi(\frac{x}{p_1p_2\ldots p_k}, p_1)$$

$$= \sum_{p_1p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k, p_1^2p_2\ldots p_k>x} p_1\psi(\frac{x}{p_1p_2\ldots p_k}, p_1)$$
$$+ \sum_{p_1^2p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k} p_1\psi(\frac{x}{p_1p_2\ldots p_k}, p_1)$$
$$= \sum_{p_1p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k, p_1^2p_2\ldots p_k>x} p_1[\frac{x}{p_1p_2\ldots p_k}]$$
$$+ O(\sum_{p_1^2p_2\ldots p_k\leq x, p_1\leq p_2\leq\ldots\leq p_k} \frac{xp_1}{p_1p_2\ldots p_k}).$$

In the sum appearing in the O-term the conditions $p_1 \leq p_2 \leq \ldots \leq p_k$ and $p_1^2p_2\ldots p_k \leq x$ imply that $p_1 \leq x^{1/(k+1)}$. Therefore this sum is

$$\leq \sum_{p_1^2p_2\ldots p_k\leq x} \frac{x^{1+1/(k+1)}}{p_1p_2\ldots p_k}$$

$$<< x^{1+1/(k+1)} \sum_{p_1 \le x} \frac{1}{p_1} \cdots \sum_{p_k \le x} \frac{1}{p_k} << x^{1+1/(k+1)} (\log\log x)^k,$$

on using the elementary estimate

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1).$$

Thus we obtain

$$\sum_{n \le x} P_k(n) = \sum_{p_1 p_2 \dots p_k \le x, p_1 \le p_2 \le \dots \le p_k} p_1 [\frac{x}{p_1 p_2 \dots p_k}] \tag{11}$$

$$+O(x^{(k+2)/(k+1)}(\log\log x)^k),$$

$$= \sum_{n p_1 p_2 \dots p_k \le x, p_1 \le p_2 \le \dots \le p_k} p_1 + O(x^{k+2)/(k+1)}(\log\log x)^k),$$

where n stands for natural numbers, and the portion of the sum containing the greatest integer function for $p_1^2 p_2 \dots p_k \le x$ is estimated trivially in exactly the same way as the O-term in (10). Two further simplifications may be made in the last sum in (11). Firstly, we may restrict n to the range $1 \le n \le x^\varepsilon$, where $0 < \varepsilon < 1$ is any fixed number. This follows from

$$\sum_{n p_1 p_2 \dots p_k \le x, n > x^\varepsilon, p_1 \le p_2 \le \dots \le p_k} p_1 << x^{1/k} \sum_{n > x^\varepsilon} n^{-1/k} \sum_{p_1 p_2 \dots p_k \le x/n} 1$$

$$<< x^{1+1/k} \sum_{n > x^\varepsilon} n^{-1-1/k} << x^{1+1/k-\varepsilon/k},$$

since the last expression is of lower order of magnitude than any term on the right-hand side of (3). Secondly, the contribution from $p_k > (x/2n)^{1/(k-1)}$ in (11) ($k \ge 2$ may be henceforth assumed to hold in view of (2)) is also negligible. Namely, from $p_k > (x/2n)^{1/(k-1)}, p_1 \le \dots \le p_k$ and $p_1 p_2 \cdots p_k \le x/n$ we have $p_1 << (x/n)^{(k-2)/(k-1)^2}$. Hence for $k > 2$

$$\sum_{n \le x^\varepsilon} \sum_{p_1 p_2 \dots p_k \le x/n, p_1 \le p_2 \le \dots \le p_k, p_k > (x/2n)^{1/(k-1)}} p_1$$

$$<< x^{1+(k-2)/(k-1)^2} \sum_{n \le x^\varepsilon} n^{-1-(k-2)/(k-1)^2}$$

$$\le x^{1+(k-2)/(k-1)^2} \zeta(1 + (k-2)/(k-1)^2) << x^{1+(k-2)/(k-1)^2},$$

on using the same argument as in (9), and for $k = 2$ we get the bound $x \log x$. Actually, even the contribution for $p_k > (x/n)^{1/k} (\log x)^D, D =$

$D(k) > 0$ sufficiently large, is also seen to be negligible by the above argument, but it will be enough to have $p_k \le (x/2n)^{1/(k-1)}$ in (11). We obtain

$$\sum_{n\le x} P_k(n) = \sum_{n\le x^\varepsilon} \sum_{p_1p_2\ldots p_k\le x/n, p_k\le (x/2n)^{1/(k-1)}, p_1\le\ldots\le p_k} p_1 + O(x^{1+\Theta}),$$

where $\Theta < 1/2$ if $k = 2$ and for $k > 2$ we have

$$\Theta = \Theta(k,\varepsilon) = \max\left(\frac{1-\varepsilon}{k}, \frac{k-2}{(k-1)^2}\right) < \frac{1}{k}.$$

Moreover, the double sum above may be written as a multiple sum, so that in fact we have

$$\sum_{n\le x} P_k(n) = O(x^{1+\Theta}) + \tag{12}$$

$$+ \sum_{n\le x^\varepsilon} \sum_{p_k\le (x/2n)^{1/(k-1)}} \sum_{p_{k-1}\le \min(p_k, \frac{x}{2^{k-2}np_k})} \cdots$$

$$\cdots \sum_{p_2\le \min(p_3, \frac{x}{2p_3\ldots p_kn})} \sum_{p_1\le \min(p_2, \frac{x}{p_2\ldots p_kn})} p_1.$$

But the conditions $p_2 \le p_3 \le \ldots \le p_k$ and $p_k \le (x/2n)^{1/(k-1)}$ imply

$$\min(p_k, \frac{x}{2^{k-2}p_kn}) = p_k, \ldots, \min(p_3, \frac{x}{2p_3\ldots p_kn}) = p_3,$$

and finally one has

$$\min(p_2, \frac{x}{p_2p_3\ldots p_kn}) = \begin{cases} p_2 & \text{if } p_2 \le (x/(p_3\ldots p_kn))^{1/2}, \\ \frac{x}{p_2p_3\ldots p_kn} & \text{if } p_2 > (x/(p_3\ldots p_kn))^{1/2}. \end{cases} \tag{13}$$

It follows that (12) becomes

$$\sum_{n\le x} P_k(n) = O(x^{1+\Theta}) + \tag{14}$$

$$+ \sum_{n\le x^\varepsilon} \sum_{p_k\le (x/2n)^{1/(k-1)}} \sum_{p_{k-1}\le p_k} \cdots$$

$$\cdots \sum_{p_3\le p_4} \sum_{p_2\le p_3} \sum_{p_1\le \min(p_2, \frac{x}{p_2p_3\ldots p_kn})} p_1.$$

Now from (14) it becomes fairly clear why (3) must hold. We have k summations over primes $p_j (1 \le j \le k)$, each of which accounts for an additional $\log x$ factor in the denominators in (3), on using (4) and (5). The summation over n will "preserve" log-factors in view of (6), and the final exponent of x in all the terms must be the same, and consequently it will be $1 + 1/k$ in view of (8) and (9). To elucidate this in more detail suppose now that in (14) we ha ve $p_k \le (x/n)^{1/k}$. Then in (13) we have the first case, since

$$p_2^2 p_3 \dots p_k \le p_k^k \le x/n.$$

Therefore the corresponding portion of the multiple sum in (14) equals

$$(15) \qquad \sum_{n \le x^\epsilon} \sum_{p_k \le (x/n)^{1/k}} \sum_{p_{k-1} \le p_k} \cdots \sum_{p_2 \le p_3} \sum_{p_1 \le p_2} p_1$$

$$\begin{aligned}
&= \sum_{n \le x^\epsilon} \sum_{p_k \le (x/n)^{1/k}} \sum_{p_{k-1} \le p_k} \cdots \sum_{p_2 \le p_3} \Big(\sum_{j=1}^{N+1} \frac{c_{1,j} p_2^2}{\log^j p_2} + O\Big(\frac{p_2^2}{\log^{N+2} p_2}\Big)\Big) \\
&= \dots = \sum_{n \le x^\epsilon} \sum_{p_k \le (x/n)^{1/k}} \Big(\sum_{j=1}^{N+1} \frac{c_{k-1,j} p_k^k}{\log^{j+k-2} p_k} + O\Big(\frac{p_k^k}{\log^{N+k} p_k}\Big)\Big) \\
&= \sum_{n \le x^\epsilon} \Big(\sum_{j=1}^{N+1} \frac{c_{k,j} (x/n)^{(k+1)/k}}{\log^{j+k-1}(x/n)} + O\Big(\frac{(x/n)^{(k+1)/k}}{\log^{N+k+1} x}\Big)\Big) \\
&= B_{k,1} \frac{x^{1+1/k}}{\log^k x} + B_{k,2} \frac{x^{1+1/k}}{\log^{k+1} x} + \dots + \\
&+ B_{k,N+1} \frac{x^{1+1/k}}{\log^{k+N} x} + O\Big(\frac{x^{1+1/k}}{\log^{k+N+1} x}\Big),
\end{aligned}$$

where $B_{k,j}$ are suitable constants $(B_{k,1} > 0)$, and where (4) and (6) was used.

The remaining portion of the multiple sum in (14), where now

$$(x/n)^{1/k} < p_k \le (x/2n)^{1/(k-1)}$$

is estimated in an analogous way, using (4)-(6). Consider the range $p_{k-1} \le p_k$ and split it into the ranges $p_{k-1} \le (x/np_k)^{1/(k-1)}$ and $(x/np_k)^{1/(k-1)} < p_{k-1} \le p_k$. If $p_{k-1} \le (x/np_k)^{1/(k-1)}$, then $p_2 \le (x/p_3 \dots p_k n)^{1/2}$, and we obtain that the corresponding portion of the multiple sum equals

$$\sum_{n \le x^\epsilon} \sum_{(x/n)^{1/k} < p_k \le (x/2n)^{1/(k-1)}} \sum_{p_{k-1} \le (x/np_k)^{1/(k-1)}} \sum_{p_{k-2} \le p_{k-1}} \cdots$$

$$\cdots \sum_{p_2 \le p_3} \sum_{p_1 \le p_2} p_1$$

$$= C_{k,1}\frac{x^{1+1/k}}{\log^k x} + C_{k,2}\frac{x^{1+1/k}}{\log^{k+1} x} + \ldots + C_{k,N+1}\frac{x^{1+1/k}}{\log^{k+N} x} + O(\frac{x^{1+1/k}}{\log^{k+N+1} x}).$$

This processes is continued, and in the j-th step $(2 \le j \le k-1)$ the range of summation for p_{k-j+1} is split into

$$p_{k-j+1} \le (\frac{x}{np_k p_{k-1}\cdots p_{k-j+2}})^{1/(k-j+1)}$$

and

$$(\frac{x}{np_k p_{k-1}\cdots p_{k-j+2}})^{1/(k-j+1)} < p_{k-j+1} \le p_{k-j+2}.$$

In the first range we have $p_{k-j} \le p_{k-j+1}, \ldots, p_2 \le p_3, p_1 \le p_2$, and this portion yields analogously as before an expression of the type appearing on the right-hand side of (3). In the second range, we continue the process of splitting up the range of summation of the next variable. All the ensuing sums will be eventually of the same type, and the last one will be

$$\sum_{n \le x^\epsilon} \sum_{(x/n)^{1/k} < p_k < (x/2n)^{1/(k-1)}} \sum_{(x/np_k)^{1/(k-1)} < p_{k-1} \le p_k}$$

$$\cdots \sum_{(x/np_3\ldots p_k)^{\frac{1}{2}} < p_2 \le p_3} \sum_{p_1 \le \frac{x}{p_2 p_3 \cdots p_k n}} p_1$$

$$= \sum_{n \le x^\epsilon} \sum_{p_k} \sum_{p_{k-1}} \cdots \sum_{(x/np_3\ldots p_k)^{\frac{1}{2}} < p_2 \le p_3} \times$$

$$\times(\sum_{j=1}^{N+1} \frac{c_{1,j}x^2}{p_2^2\cdots p_k^2 n^2 \log^j(\frac{x}{p_2\cdots p_k n})} + O(\frac{x^2}{p_2^2\cdots p_k^2 n^2 \log^{N+2}(\frac{x}{p_2\cdots p_k n})})).$$

This will eventually reduce again to an expression analogous to the one on the right-hand side (3), since the terms coming from both limits of summation for $p_2, \ldots, p_{k-1}$ will at the end give rise to the same type of terms. Take, for example, the lower limits of summation. Using (5) it is seen that summation over $p_2 > (x/np_3\ldots p_k)^{1/2}$ (i.e. the lower limit) gives rise to terms of the type

$$x^{3/2}n^{-3/2}p_3^{-3/2}\ldots p_k^{-3/2}\log^{-j-1}(x^{3/2}\ldots p_k^{-3/2}) \ (j = 1, \ldots, N+2),$$

and in general summation over $p_m > (x/(np_{m+1}\cdots p_k))^{1/2}(2 \le m \le k-1)$ gives rise to terms of the type

$$x^{(m+1)/m} n^{-(m+1)/m} p_{m+1}^{-(m+1)/m} \cdots$$

$$\cdots p_k^{-(m+1)/m} \log^{-j-m+1}(x^{(m+1)/m} \cdots p_k^{-(m+1)/m}).$$

Finally summation over p_k gives

$$(16) \quad \sum_{n \le x^\epsilon} x^{(k+1)/k} n^{-(k+1)/k} \Big(\sum_{j=1}^{N+1} D_{k,j} \log^{1-k1j}(x/n) + O(\log^{-k-N-1} x)\Big)$$

$$= x^{(k+1)/k} \Big(\sum_{j=1}^{N+1} E_{k,j} \log^{1-k-j} x + O(\log^{-k-N-1} x)\Big)$$

with suitable constants $D_{k,j}$ and $E_{k,j}(E_{k,1} > 0)$, since the upper limit of summation $(x/2n)^{1/(k-1)}$ for p_k gives by (5) an expression absorbed in the error term in (3) and (16). Likewise the upper limits of summation for $p_2, p_3, \ldots, p_{k-1}$ will eventually yield again expression of the type (16), so collecting all such expressions it is seen that (3) is established. The constants $A_{k,j}(j = 1, \ldots, N)$ in (3) can be certainly written down explicitly in closed form, although the above proof would lead to fairly cumbersome expressions for these constants. It may be also remarked that our theorem gives easily the asymptotic formula

$$(17) \quad \sum_{n \le x} (B(n) - P_1(n) - \ldots - P_{k-1}(n))$$

$$= x^{1+1/k} \Big(\sum_{j=1}^{N+1} A_{k,j} \log^{1-j-k} x + O(\log^{-k-N-1} x)\Big),$$

where $k \ge 2$ and the hypotheses are the same as in (3). Here $B(n) = \sum_{p^\alpha \| n} \alpha p$ is the sum of all prime divisions of n. This formula, with the right-hand side as in (1), was given by Alladi and Erdös [2]. One obtains easily (17) from (3) when one observes that the sum in (17) is empty for $\Omega(n) \le k-1$, and that for $r = \Omega(n) \ge k$ one has

$$(18) \quad B(n) - P_1(n) - \ldots - P_{k-1}(n) = P_k(n) + \ldots + P_r(n)$$

$$= P_k(n) + O(P_{k+1}(n) \log n),$$

since trivially $\Omega(n) << \log n$. Hence summing (18) with the aid of (3) we obtain (17).

By the same method we sould evaluate asymptotically the sum $\sum_{n\leq x}(P_k(n))^c$, where $c > 0$ is a fixed real number, and $k \geq 1$ is a fixed integer. The result would be an expression similar to the right-hand side of (3), with $x^{1+1/k}$ replaced by $x^{1+c/k}$, and $A_{k,j} = A_{k,j}(c)$. The case $c < 0$ requires new methods. For $c = -1$ (sum of reciprocals, where $n = 1$ is excluded from summation and n with $\Omega(n) < k$) this problem was solved in [5] and [6].

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REZIME

O k - TOM PROSTOM FAKTORU PRIRODNOG BROJA

Neka je $P_k(n)$ k-ti najveći prosti faktor prirodnog broja $n \geq 1$ ako n ima bar k prostih faktora, a neka je $P_k(n) = 0$ u suprotnom. Dobivena je asimptotska formula za sumu $\sum_{n \leq x} P_k(n)$.

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