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DISTANCE PRESERVING MAPS ON ABELIAN LATTICE ORDERED GROUPS

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Abstract

If f is a map from an Abelian lattice ordered group G_1 (endowed with a root function γ_2 of order two) onto an Archimedean Abelian lattice ordered group G_2 with $f(0) = 0$ and $|f(x) - f(y)|$ depends functionally on $|x - y|$, then f is additive.

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Mazur and Ulam [7] have proved that every isometry of a normed real vector space onto a normed vector space is linear up to translation. This result was extended in many different directions. Vogt [12] has replaced isometries by the more general maps with the property that the distance between image points depends functionally on the distance between domain points.

Swamy [10],[11] has defined an isometry in an Abelian lattice ordered group G as a bijection $f : G \to G$ with the property

$$|f(x) - f(y)| = |x - y| \quad (x, y \in G)$$

(without bijectivity by Jakubik [10] under the name weak isometry). Swamy [10] has proved that every isometry on a lattice ordered group G is of the form $T(x)+a$ where a is a fixed element of G and T is an involutory isometric group automorphism of G.

We shall investigate, in this paper, maps from Abelian lattice ordered group G_1 into an Archimedean Abelian lattice ordered group G_2 which preserves the equlity of distance in the sense of autometrized spaces. We shall prove that the surjective distance preserving map f with $f(0) = 0$ is additive.

1. Let G be an Abelian group written additively with a neutral element 0. G is an Abelian lattice order group if G is also a lattice under a partial ordered relation $\leq$ with the property that $a \leq b$ implies $c + a \leq c + b$ for all $c \in G$. A partially ordered group G is said to be Archimedean if $a > 0$ and $b > 0$ than $na > b$ for a suitable $n \geq 1$, where $na = \underbrace{a + \cdots + a}_{n}$. We extract the following known result (see [2], footnote on page 12).

Proposition 1. *If G is an Archimedean Abelian lattice ordered group, then $na < b$ $(n = 0, \pm 1, \pm 2, \ldots)$ implies $a = 0$.*

Let $G^+ = \{x : x \in G, x \geq 0\}$. The absolute $|a|$ of an element $a \in G$ is defined by $|a| = a \bigvee (-a)$. It has the following properties:

(i) $|a| \in G^+$, i.e. $|a| \geq 0$, for all $a \in G$ and the equality holds if and only if $a = 0$;

(ii) $|-a| = |a|$;

(iii) $|a + b| \leq |a| + |b|$;

(iv) $|na| = n|a|$.

Remark. The property (iii) holds only for Abelian groups. The properties (i), (ii) and (iv) hold also for the noncommutative case.

Let G_1 and G_2 be lattice ordered Abelian groups.

Definition 1. *A map $f : G_1 \to G_2, f(0) = 0$ preserves the equality of distance if there exists a function $p : G_1^+ \to G_2^+$ such that for each x and y from G*

$$|f(x) - f(y)| = p(|x - y|).$$

The function p is called the gauge function for f.

Theorem 1. *Let G_1 and G_2 be Abelian lattice ordered groups.If f is a map from G_1 into G_2 with $f(0) = 0$ then the following statements are equivalent:*
a) f preserves the equality of distance;
b) whenever x, y, z and u are in G_1 and $|x - y| = |z - u|$,then

$$|f(x) - f(y)| = |f(z) - f(u)|;$$

If $G_1 = G_2$ then the above assertions are equivalent to the following one:
c) there exists an integer n such that

$$| f(x) - f(y) |= n \mid x - y \mid \quad (x, y \in G_1).$$

Proof. a) implies b). Let x, y, z and u be from G_1 such that $|x - y| = |z - u|$. Then we have

$$|f(x) - f(y)| = p(|x - y|) = p(|z - u|) = |f(z) - f(u)|.$$

b) implies a). We define $p(z) = |f(z)|$ $(z \in G_1^+)$. Then p is a gauge function for f. We have for any $x, y \in G_1$ and $z = |x - y|$,

$$|x - y| = z = |z| = |z - 0|.$$

Therefore, by b) we obtain

$$p(|x - y|) = p(z) = |f(z)| = |f(x) - f(y)|.$$

Let $G_1 = G_2$. a) implies c). Let $d(x, y) = p(|x - y|)$. Then $d(x, y)$ is a translation invariant map, *i.e.* $d(a, b) = d(a + c, b + c) = d(c + a, c + b)$ and symmetric map, *i.e.*$d(a, b) = d(b, a)$. Hence d is an intrinsic metric in the sense of Holland [3]. By the corollary from [3], there exists an integer n such that

$$p(|x - y|) = d(x, y) = n|x - y|.$$

c) implies a). If a map f satisfies c) then a gauge function for f is $p(z) = nz$, $(z \in G_1^+)$.

Remark. By the preceding theorem the gauge function p for a map $f : G \to G$ which preserves the equality of distance has to be of the form $p(z) = nz$, for an integer n. We shall call $n|x - y|$ an intrinsic n- metric. Jakubik [1] and [2] has defined an n-isometry on a lattice ordered group as a function from G onto G which preserves the intrinsic n-metric.1-isometries are always n-isometries for every n. For Abelian lattice ordered group every n-isometry is also an 1-isometry.

2. Using the ideas from [1] and [12] we shall prove the following theorem.

Theorem 2. *Let G be an Archimedean Abelian lattice ordered group. Let A be a bounded subset of G, i.e. there exists an element b from G such that $|x| \leq b$ for all x from A. Suppose there exists an element $a \in A$, a surjective isometry (congruence in the sense of Swamy [10]) $g : A \to A$ and a natural number m such that for all $x \in A$*

(1) $\quad m|a - x| \leq |g(x) - x|.$

Then, every surjective isometry $h : A \to A$ fixes a.

Proof . Since each isometry in an Abelian lattice ordered group G is an injection ($x \neq y$ implies $|x - y| \neq 0$, hence $|f(x) - f(y)| \neq 0$ and so $f(x) \neq f(y)$), h^{-1} and g^{-1} exist. We have that h, g, h^{-1} and g^{-1} are bijective isometries. Hence any finite composition of them is also a bijective isometry. We define a sequence $\{g_n\}$ of isometries on A in the following way:

$$g_1 = g,\ g_2 = hg_1h, \ldots,\ g_{n+1} = g_{n-1}g_n(g_{n-1})^{-1}.$$

We also define a sequence $\{a_n\}$ from G in the following way:

$$a_1 = a, a_2 = h(a), \cdots, a_{n+1} = g_{n-1}(a_n) \quad (n \geq 2).$$

Starting from (1) we obtain, by induction,

(2) $\quad m|a - x| \leq |g_n(x) - x| \quad (x \in A).$

Taking $x = a_{n+1}$ in (2), we have

$$m|a_{n+1} - a_n| = m|a_n - a_{n+1}| \leq |g_n(a_{n+1}) - a_{n+1}| = |a_{n+2} - a_{n+1}|.$$

Hence, by induction, we obtain

(3) $\quad m^n|a_2 - a_1| \leq |a_{n+2} - a_{n+1}| \quad (n \in N).$

Since A is ordered bouned, we have for all $n \in N$

$$|a_{n+2} - a_{n+1}| \leq |a_{n+2}| + |a_{n+1}| \leq 2b.$$

Hence by (3), we obtain

(4) $n|a_2 - a_1| \leq m^n|a_2 - a_1| \leq 2b$ for each $n \in N$.

Since G is an Archimedean lattice ordered group, (4) implies by Proposition 1. $|a_2 - a_1| = 0$. Hence,

$$a = a_1 = a_2 = h(a).$$

This completes the proof.

A function $\gamma_n : G \to G$ is a root function of order $n \in N \cup \{0\}$ on an Abelian group G if

(a) $\gamma_n(x + y) = \gamma_n(x) + \gamma_n(y)$;

(b) $n\gamma_n(x) = x$ (see [8]);
hold.

Proposition 2. *Let G be an Abelian lattice ordered group. Then, for each non-negative integer n there exists at most one root function γ_n on G.*

Proof. Since every lattice ordered group is isolated (see E in 5.1 from [2]) it is also torsion-free. Hence, for any $n \in N \cup \{0\}$ there exists at most one root function γ_n of order n.

Now, we have the main result of this paper.

Theorem 3. *Let G_1 and G_2 be Abelian lattice ordered groups such that (G_1 has a root function γ_2) G_2 is Archimedean.Let $f : G_1 \to G_2$ with $f(0) = 0$ be a surjective map which preserves the equality of distance.Then, f is additive.*

Proof. Let x be an arbitrary but fixed element from G_1. We define

$$A = \{y : y \in G_2 \text{ and } \quad |y| = |2f(x) - y| \leq 2|f(x)|\}$$

and $g : A \to A$ by $g(y) = 2f(x) - y$. It is obvious that g is an isometry from A onto A. Taking $a = f(x)$, we have

$$2|a - y| = 2|f(x) - y| = |(2f(x) - y) - y| = |g(y) - y|.$$

Hence, g satisfies condition (1) for $m = 2$ from Theorem 2. Since the set A is bounded , we can apply Theorem 2. Therefore, we obtain that every surjective isometry of A fixes a. Now we can define an appropriate isometry h. Let $u = f^{-1}(2f(x))$. We define $h : A \to A$ by

$$h(y) = f(u - f^{-1}(y)).$$

Now, we can prove that h is well-defined. If $f(x_1) = f(x_2) = y$ then we have

$$|f(u - x_1) - f(u - x_2)| = p(|(u - x_1) - (u - x_2)|) = p(|x_2 - x_1|) =$$
$$= |f(x_2) - f(x_1)| = |y - y| = 0.$$

Hence $f(u - x_1) = f(u - x_2)$.

If $f(x_1) = y_1$ and $f(x_2) = y_2$, then

$$|h(y_1) - h(y_2)| = |f(u - x_1) - f(u - x_2)| =$$
$$p(|x_2 - x_1|) = |f(x_2) - f(x_1)| = |y_2 - y_1|,$$

i.e. h is an isometry. We shall prove that h is an isomerty from A into A. Namely, if $y_1 \in A$ and $y_1 = f(x_1)$, then

$$|2f(x) - h(y_1)| = |2f(x) - f(u - x_1)| = |f(u) - f(u - x_1)| =$$
$$p\,(|x_1 - 0|) = |f(x_1) - f(0)| = |y_1 - 0| = |y_1| = |2f(x) - y_1| =$$
$$|f(u) - f(x_1)| = p\,(|u - x_1|) = p\,(|(u - x_1) - 0|) =$$
$$|f(u - x_1) - f(0)| = |h(y_1) - 0| = |h(y_1)|\,,$$

i.e. $h(y_1) \in A$. Since h is its own inverse, h is a surjective isometry from from A onto A.

Now, we shall prove the following equality

(5) $\qquad f(2x) = 2f(x) \quad (x \in G_1)$. Since by Theorem 2 h fixes a and

$$f(x) = a = h(a) = f(u - f^{-1}(a)) = f(u - f^{-1}(f(x)))$$

(we shall use also that $u = f^{-1}(2f(x))$ we obtain

$$|f(2x) - 2f(x)| = |f(2x) - f(u)| = p\,(|2x - u|) =$$
$$p\,(|x - (u - x)|) = |f(x) - f(u - x)| = 0.$$

Hence (5).

We define for an arbitrary but fixed $y \in G_1$

$$F_y(x) = f(x+y) - f(y) \quad (x \in G_1).$$

It is obvious that $F_y(0) = 0$ and that F_y is surjective. Since

$$|F_y(x_1) - F_y(x_2)| = |f(x_1+y) - f(x_2+y)| =$$

$$p\left(|(x_1+y) - (x_2+y)|\right) = p\left(|x_1 - x_2|\right).$$

F_y preserves the equality of distance. By Proposition 2 there exists a root function γ_2 of order 2 on G_1. Hence, by (5) it holds that

$$F_y(z) = 2F_y(\gamma_2(z)) \quad (z \in G_1).$$

This equality implies that for any x and y from G_1 it holds that

$$f((x-y)+y) - f(y) = F_y(x-y) = 2F_y(\gamma_2(x-y)) =$$

$$2[f(\gamma_2(x-y)+y) - f(y)].$$

Hence,

$$f(x) + f(y) = 2f(\gamma_2(x+y)) = f(x+y).$$

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REZIME

PRESLIKAVANJA KOJA OČUVAVAJU RASTOJANJE NAD ABELOVIM MREŽASTIM GRUPAMA

U radu je dokazano da je svako preslikavanje f sa komutativne mrežaste grupe G_1 na Arhimedovsku komutativnu mrežastu grupu G_2 sa osobinama da $|f(x) - f(y)|$ funkcionalno zavisi od $|x - y|$ i $f(0) = 0$ uvek aditivno.

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