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ASYMPTOTICS OF NONOSCILLATORY SOLUTIONS OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract

Let $f(x)$ be continuous for $x > 0$; we consider the existence and asymptotic behavior for $x \to \infty$ of nonoscillatory solutions of the equation

$$y'' + f(x)y = 0. \tag{0.1}$$

In our analysis there are neither hypotheses concerning the sign of the function $f(x)$ nor hypotheses concerning the absolute integrability of $f(x)$ over (a, ∞). First we prove a sufficient condition for the existence of nonoscillatory solutions of (1.1) and then obtain asymptotic formulae for such solutions.

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1. Introduction

Result similar to the ones presented here are given by A. R. Its [1] and by J. Mahony [2]. Both Its and Mahony consider the asymptotic behavior of oscillatory solutions of (0.1) as well as the asymptotic behavior of nonoscillatory solutions. Its assumes

$$f(x) = x^{\beta} P(x^{1+\alpha}) + cx^{-2},$$

where P is a smooth, periodic function of period 1 such that

$$\int_0^1 P(t)dt = 0,$$

with α, β, and c real numbers, and with

$$\beta = \alpha - 1 \text{ and } \int_0^1 x^{-2}\{\int_0^x tP(t)dt\}^2 dx < (c + 1/4)(1 + \alpha)^2.$$

Mahony sketches at the end of his paper how his results for the interesting case where $f(x) = \sin(ax)/x$ can be extended to cover the cases in which

$$f(x) = \sin(ax)/x + \sum_2^{\infty} b_r x^{-r}$$

and in which

$$f(x) = p(x)/x,$$

where p is a periodic function of period τ, again of zero mean.

Our results neither contain nor are contained in the result of Its and Mahony. In particular the condition in both Its and Mahony concerning the zero mean of a periodic part of the coefficient function f is not needed in our reasonings.

Regarding the existence of nonoscillatory solutions, we refer to Willett's paper, [3], where sufficient conditions are obtained for this. We are primarily interested in asymptotics and our conditions imply results pertinent to that end. Note also the conditions in our nonoscillation theorem are simple and easy to verify.

We use successive approximations in our proofs. Note we are only concerned with one solution of equation (0.1) and the construction of a second linearly independent solution follows by usual Wronskian techniques.

2. Results

First we prove existence of nonoscillatory solutions for equation (0.1).

Theorem 1. *Let*

$$h(x) = \int_x^\infty f(x)dt \tag{2.1}$$

and let there exist a positive continuous function g, and an $x_0 > 0$, such that $g(x) \to 0$ as $x \to \infty$, while

$$|h(x)| \le g(x), \quad x \ge x_0, \tag{2.2}$$

and

$$\int_x^\infty g^2(t)dt \le cg(x), \quad x \ge x_0, \tag{2.3}$$

with c a constant such that

$$0 < c < 1/4. \tag{2.4}$$

Then the equation (0.1) has nonoscillatory solutions for $x \ge x_0$.

Proof. We prove first that the integral equation

$$u(x) = -\int_x^\infty (u(t) - h(t))^2 dt \tag{2.5}$$

has a solution $z(x)$ such that

$$z(x) = O(g(x)), \quad x \to \infty. \tag{2.6}$$

We shall use this z to construct nonoscillatory solutions of (0.1). Introduce a sequence $\{z_n(x)\}$, defined for $x \ge x_0$, as

$$z_0(x) = 0, \quad z_n(x) = -\int_x^\infty (h(t) - z_{n-1}(t))^2 dt, \quad n = 1, 2, \dots . \tag{2.7}$$

First we show that

$$|z_n(x)| < 4cg(x), \quad x \ge x_0. \tag{2.8}$$

Obviously inequality (2.8) is valid for $n = 1$ since, by use of hypotheses (2.2) and (2.3),

$$|z_1(x)| = \int_x^\infty h^2(t)dt \le \int_x^\infty g^2(t)dt \le cg(x), \quad x \ge x_0.$$

If we suppose that (2.8) holds for $n = k$ then, applying (2.2), (2.3) and (2.4), we have

$$|z_{n+1}(x)| \le \int_x^\infty (z_n(t)| + |h(t)|)^2 dt < (4c+1)^2 \int_x^\infty g^2(t)dt < 4cg(x)$$

for $x \geq x_0$. Hence, by induction, (2.8) holds for all integers $n \geq 0$.

Next we prove that $\{z_n(x)\}$ is a uniformly convergent sequence of continuous functions. Continuity follows from the definition of $z_n(x)$ and the continuity of $h(x)$. In order to show the convergence, we prove the inequality

$$|z_{n+1}(x) - z_n(x)| \leq \frac{(4c)^{n+1}}{4} g(x), \quad n = 0, 1, \ldots \quad . \tag{2.9}$$

for $x \geq x_0$. By (2.2), (2.3) and the definition of $z_0(x)$, inequality (2.9) is valid for $n = 0$. Assume, for $x \geq x_0$, that (2.9) holds for $n = k - 1$, so

$$|z_k(x) - z_{k-1}(x)| \leq \frac{(4c)^k}{4} g(x); \tag{2.10}$$

then one gets from

$$|z_{k+1}(x) - z_k(x)| \leq \int_x^\infty |z_k(t) - z_{k-1}(t)|(|z_k(t)| + |z_{k-1}(t)| + 2|h(t)|)dt,$$

and use of (2.2), (2.3), (2.4), (2.8) and (2.10), that

$$|z_{k+1}(x) - z_k(x)| < \int_x^\infty \frac{(4c)^k}{4}(8c + 2)g^2(t)dt < \frac{(4c)^{k+1}}{4} g(x)$$

for $x \geq x_0$, which proves (2.9) for all integers $n \geq 0$. Since

$$z_{n+1}(x) = \sum_{k=0}^{n} (z_{k+1}(x) - z_k(x)), \tag{2.11}$$

we conclude from (2.9) that $\{z_n(x)\}$ converges uniformly on $[x_0, \infty)$ and the function

$$z(x) = \lim_{n \to \infty} z_n(x) \tag{2.12}$$

is continuous. Also, from (2.9) and (2.11) there follows, for $x \geq x_0$,

$$|z_{n+1}(x)| \leq \frac{g(x)}{4} \sum_{k=0}^{n} (4c)^{k+1} < \frac{c}{1 - 4c} g(x). \tag{2.13}$$

The appraisal of (2.6) now follows at once.

We are left with the proof that $z(x)$ is a solution of (2.5). Since, by (2.2) and (2.8),

$$(z_n(x) - h(x))^2 < (4c + 1)^2 g^2(x)$$

for $x > x_0$, and the integral on the left side of (2.3) is convergent, we get, using the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_x^\infty (z_n(t) - h(t))^2 dt = \int_x^\infty (z(t) - h(t))^2 dt,$$

and it follows that $z(x)$ is a solution of equation (2.5).

Finally, to construct a solution of (0.1), we introduce the function $\xi(x)$ by

(2.14) $$\xi'(x)/\xi(x) = h(x) - z(x), \quad x \geq x_0.$$

By a straightforward calculation we show that $\xi(x)$ satisfies the equation

$$\xi''(x) + \{z'(x) - (z(x) - h(x))^2 - h'(x)\}\xi(x) = 0$$

for $x \geq x_0$. Noting that $z(x)$ is a solution of equation (2.5) and that, by (2.1),

$$h'(x) = -f(x),$$

we conclude that $\xi(x)$ is a solution of (0.1). If we integrate (2.14) over (a, x), with $a \geq x_0$, we obtain a solution of (0.1) given by

(2.15) $$\xi(x) = \xi_0 \exp\{\int_a^x (h(t) - z(t))dt\},$$

which is obviously a nonoscillatory one. This proves Theorem 1.

Now we give a theorem concerning the asymptotic behavior of the solution of (0.1) just constructed.

Theorem 2. *Let the functions h and g be as defined as in Theorem 1. If*

(2.16) $$\int_x^\infty g^2(t)dt \leq c(x)g(x), \quad x \geq x_0,$$

where c is a monotonely decreasing functions such that

(2.17) $$0 < c(x) \leq x < 1/4,$$

and if, for some positive integer n,

(2.18) $$\int_0^\infty c^n(x)g(x)dx < \infty,$$

then the solution (2.15) of (0.1) satisfies the asymptotic equality

(2.19) $$y(x) \sim A \exp(\int_a^x (h(t) - z_{n-1}(t))dt), \quad x \to \infty,$$

where A = constant and the z_n are defined by (2.7).

Proof. Notice that all hypotheses of Theorem 1 hold. For the sequence $z_n(x)$ put

$$r_n(x) = \sum_{k=n}^{\infty}(z_k(x) - z_{k-1}(x)). \tag{2.20}$$

For the function z defined by (2.12) we have

$$z(x) = z_{n-1}(x) + r_n(x). \tag{2.21}$$

In the same manner as in the proof of inequality (2.9) it can be shown (by use of the fact the c is monotonely decreasing) that

$$|z_k(x) - z_{k-1}(x)| \le \frac{(4c(x))^k}{4} g(x), \quad k = 1, 2, \dots , \tag{2.22}$$

for $x \ge x_0$. Now, from (2.17), (2.20) and (2.22), there follows

$$|r_n(x)| \le \frac{g(x)}{4} \sum_{k=n}^{\infty}(4c(x))^k = \frac{4^{n-1}c^n(x)}{1 - 4c(x)} g(x) \le \frac{4^{n-1}}{1 - 4c} c^n(x) g(x). \tag{2.23}$$

But condition (2.18) shows that the integral

$$\int_a^{\infty} r_n(x)dx$$

converges absolutely and, by (2.15) and (2.21), we obtain the asymptotic behavior given by (2.19).

3. Examples

Example 1 Let

$$f(x) = \frac{a + b\sin x}{x^{\alpha} \ln^{\beta} x}, \quad \alpha \ge 1.$$

We shall treat two cases and obtain asymptotic formulae which hold for $x \to \infty$. For the first case suppose $a \ne 0$ while b is an arbitrary constant. Now, for large x,

$$h(x) = \frac{a}{\alpha - 1} x^{1-\alpha} \ln^{-\beta} x + O(x^{1-\alpha} \ln^{-\beta-1} x),$$

for $\alpha > 1$ and β arbitrary. For any fixed $\epsilon > 0$ we take

$$g(x) = \frac{|a| + \epsilon}{\alpha - 1} x^{1-\alpha} \ln^{-\beta} x.$$

Thus, for large x ,

$$\int_x^\infty g^2(t)dt = \frac{(|a|+\epsilon)^2}{(2\alpha-3)(\alpha-1)^2} x^{3-2\alpha}\ln^{-2\beta} x + O(x^{3-2\alpha}\ln^{-2\beta-1} x).$$

If $\alpha > 2$, assumptions (2.3) and (2.4) hold, for any a, b and β. Moreover, since there exists a constant $M > 0$ such that the function

$$c(x) = Mx^{2-\alpha}\ln^{-\beta} x$$

fulfills the conditions of Theorem 2, (with (2.18) being satisfied for $n = 1$), we conclude

$$y(x) \sim A\exp(\int_a^x h(t)dt).$$

Since

$$\int_a^\infty h(x)dx < \infty,$$

the solution tends to a nonzero constant.
If $\alpha = 2$ and $\beta > 0$ then we take $c(x) = M\ln^{-\beta} x$ and note

$$\int_a^\infty c^n(x)g(x)dx = \int_a^\infty O(x^{-1}\ln^{-(n+1)\beta} x)dx$$

converges for sufficiently large n. If $\beta > 1/2$ we take $n = 1$ in (2.18) and get the following results:

$$\text{if } \beta > 1, \quad y \sim \text{ constant };$$

$$\text{if } \beta = 1, \quad y \sim A(\ln x)^a;$$

$$\text{if } \frac{1}{2} < \beta < 1, \;\; y \sim A\exp(\frac{a}{1-\beta}\ln^{1-\beta} x).$$

If $1/3 < \beta \leq 1/2$ condition (2.18) is fulfilled for $n = 2$ and one gets,

$$\text{if } \beta = 1/2, \;\; y \sim A\exp(2a\ln^{1/2} x)(\ln x)^{a^2};$$

$$\text{if } 1/3 < \beta < 1/2, \; y \sim A\exp(\frac{a}{1-\beta}\ln^{1-\beta} x)\exp(\frac{a^2}{1-2\beta}\ln^{1-2\beta} x),$$

and so on; the asymptotic behavior can be obtained for any positive β.

If $\alpha = 2$, and $\beta = 0$ the conditions of Theorem 1 are fulfilled for $|a| < 1/4$ and we have the existence of nonoscillatory solutions of our equations, but we cannot find a function $c(x)$ satisfying condition (2.18) of Theorem 2. Thus in this case we are not able to obtain the asymptotic behavior of solutions. This ends our discussion of the first case.

For the second case suppose $a = 0$. Now we have, for large x

$$h(x) = bx^{-\alpha}(ln^{-\beta}x)\cos x + O(x^{-\alpha-1}\ln^{-\beta}x),$$

$$g(x) = (|b| + \epsilon)x^{-\alpha}\ln^{-\beta}x,$$

$$\int_x^\infty g^2(t)dt = \frac{(|b|+\epsilon)^2}{2\alpha-1}x^{1-2\alpha}\ln^{-2\beta}x + O(x^{1-2\alpha}\ln^{-2\beta-1}x),$$

$$c(x) = Mx^{1-\alpha}\ln^{-\beta}x.$$

If $\alpha > 1$, the conditions of Theorem 2 hold for any β, and if we take $n = 1$ in (2.18) it is easily seen that the solution tends to a nonzero constant as $x \to \infty$. If $\alpha = 1$ and $\beta > 1/2$, the situation is the same. If $\alpha = 1$ and $\beta = 1/2$, take $n = 2$ in (2.18) and get

$$y \sim A(\ln x)^{b^2/2}.$$

If $\alpha = 1$ and $1/3 < \beta < 1/2$ take $n = 2$ in (2.18) and get

$$y \sim A\exp(\frac{b^2}{2(1-2\beta)}\ln^{1-2\beta}x);$$

if $\alpha = 1$ and $1/4 < \beta \leq 1/2$, take $n = 3$ in (2.18) and get

$$y \sim A\exp(b^2\ln^{1/2}x)(\ln x)^{b^4/4}.$$

If $\alpha = 1$ and $1/5 < \beta < 1/4$ take $n = 4$ in (2.18) and get

$$y \sim A\exp(\frac{b^2}{2(1-\beta)}\ln^{1-2\beta}x)\exp(\frac{b^4}{4(1-4\beta)}\ln^{1-4\beta}x).$$

If $\alpha = 1$ and $1/6 < \beta \leq 1/5$, thte behavior is the same as above, although one has to take $n = 5$.

If $\alpha = 1$ and $\beta = 0$ our method gives us only the existence of nonoscillatory solutions (and not the asymptotic behavior) for $|b| < 1/4$. However, it is shown by Its that in this case solutions are nonoscillatory for $|b| < 1/\sqrt{2}$ and the asymptotic behavior is also determined.

Example 2 Let

$$f(x) = ax^{\alpha}\cos(e^{\beta x}), \quad a \neq 0, \beta > 0.$$

Notice that, if $\alpha > 0$,

$$\sup_{x \geq x_0} f(x) = -\inf_{x \geq x_0} f(x) = \infty.$$

By partial integration we get

$$h(x) = -\frac{a}{\beta}x^{\alpha}e^{-\beta x}\sin(e^{\beta x}) - \frac{a}{\beta}\int_x^{\infty}(\alpha t^{\alpha-1} - \beta t^{\alpha})e^{-\beta t}\sin(e^{\beta x})dt;$$

there exists positive constants M_1, M_2 such that

$$g(x) = M_1 x^{\alpha}e^{-\beta x}$$

with

$$\int_x^{\infty} g^2(t)dt < c(x)g(x),$$

where

$$c(x) = M_2 x^{\alpha}e^{-\beta x}.$$

Therefore, Theorem 2 holds (take $n = 1$ in (2.18)) and it follows that (0.1) has a nonoscillatory solution which tends to a nonzero constant for any a, α and $\beta > 0$.

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REZIME

ASIMPTOTIKA NEOSCILATORNIH REŠENJA LINEARNE DIFERENCIJALNE JEDNAČINE DRUGOG REDA

Neka je $f(x)$ neprekidna funkcija za $x > 0$. U radu je razmatrana egzistencija i asimptotsko ponašanje za $x \to \infty$ neoscilatornog rešenja jednačine

$$y'' + f(x)y = 0.$$

Pri ovoj analizi ne koriste se pretpostavke kako o znaku funkcije $f(x)$ tako ni o apsolutnoj integrabilnosti funkcije $f(x)$ nad (a, ∞). Prvo je dobijen dovoljan uslov za postojanje neoscilatornih rešenja navedene diferencijalne jednačine, a zatim asimptotska formula za ova rešenja.

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