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QUASIASYMPTOTIC EXPANSIONS AND THE STIELTJES TRANSFORMATION

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Abstract

Using the notation of the quasiasymptotic expansion of a tempered distribution at ∞ an ordinary asymptotic expansion of the Stieltjes transformation of function from $L^1_{loc}(0,\infty)$ is obtained with the ordinary, asymptotic expansion of the form $\sum_{i=1}^{\infty} \frac{a_i}{t^i}$, $t \to \infty$.

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1. Notations and known results

Denote by S the space of rapidly decreasing smooth functions defined on the real line $\mathbf{R}$, supplied with the usual topology. Its dual, the space of tempered distributions is S' and S'_+ is its subspace with elements supported by $[0,\infty)$.

Recall [4] a continuous positive function L defined on $(a,\infty), a > 0$, is slowly varying at infinity if for $\lambda > 0$

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1$$

We denote by $\sum_\infty$ the set of all the slowly varying (in short sv) function at ∞.

The quasiasymptotic behaviour and quasiassymptotic expansion at ∞ of an $f \in S'_+$ were studied by Vladimirov, Drožinov and Zavialov [1]. In [3] Pilipović gave a slight modification of the quasi asymptotic expansion at ∞.

Recall, $f \in S'_+$ has the quasiasymptotic behaviour at ∞ with respect to $k^\nu L(k)$, $\nu \in \mathbf{R}$, $L \in \sum_\infty$, with the limit $g \in S'$ if

$$\lim_{k\to\infty} < \frac{f(kt)}{k^\nu L(k)}, \varphi(t) >=< g(t), \varphi(t) >, \varphi \in S,$$

$<.,.>$ is the dual pairing between S' and S. In this case we write $f \overset{q}{\sim} g$ at ∞ with respect to $k^\nu L(k)$.

Let us recall that the family of homogeneous distributions $f_{\nu+1}$, $\nu \in \mathbf{R}$, is defined by

$$f_{\nu+1}(t) = \begin{cases} \frac{H(t)t^\nu}{\Gamma(\nu+1)}, & \nu > -1, \\ f^{(n)}_{\nu+n+1}(t), & \nu \le -1,\ n+\nu > -1,\ n \in \mathbf{N}, \end{cases}$$

where H is Heaviside's function; Γ is the gamma function.

Let $\nu \in \mathbf{R}$, $A > 0$, and $L \in \sum_\infty$. We put

$$f_{L,A,\nu+1}(t) = \begin{cases} \frac{H(t-A)t^\nu L(t)}{\Gamma(\nu+1)} & \nu > -1, \\ f^{(n)}_{L,A,\nu+n+1}(t), & \nu \le -1,\ n+\nu > -1,\ n \in \mathbf{N}, \end{cases}$$

where n is the smallest integer for which $n+\nu > -1$. This definition includes, for example, distribution $\delta^{(k)}(t-a)$, $a > 0$, $(H(t-a)\ln t)^{(k)}$, $a > 0$, $k \in \mathbf{N}$. For $A = 0$ we use the notation $f_{L,\nu+1}$.

We have

$$f_{L,A,\nu+1}(t) \overset{q}{\sim} f_{\nu+1}(t), t \in \mathbf{R}, \text{ at } \infty \text{ with respect to } k^\nu L(k).$$

Let us denote by Λ the set N or the set of the form $\{1, 2, \ldots, N\}$, $N \in \mathbf{N}$. In the second case we shall also use the symbol Λ_N.

We say [2] that an $f \in S'_+$ has the quasiaysmptotic expansion at ∞ with respect to $\{(k^{\nu_i} L_i),\ i \in \Lambda\}$, $L_i \in \sum_\infty$, if there are complex numbers

$a_i \neq 0$ and $A_i \in \mathbf{R}$, $i \in \Lambda$, such that for any $m \in \Lambda$

$$\frac{f(kt) - \sum_{i=1}^{m} a_i f_{L_i,A_i,\nu_i+1}(kt)}{k^{\nu_m} L_m(k)} \to 0 \text{ in } S'.$$

In that case we write

(1) $$f \overset{q.e}{\sim} \sum_{i\in\Lambda} a_i f_{L_i,A_i,\nu_i+1} \text{ at } \infty \text{ with respect to } \{(k^{\nu_i} L_i),\ i \in \Lambda\}.$$

The Stieltjes transformation of distribution was considered in [2].

Let $f \in S'$. We say that $f \in \mathcal{J}'(r)$, $r \in \mathbf{R}\backslash(-\mathbf{N})$ if there exits an $m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and a locally integrable function F such that

(2) $$\text{a) } f = F^{(m)}.$$

$$\text{b) } \int_{-\infty}^{+\infty} | F(t)(t+z)^{-r-m-1} | dt < \infty, \text{ for } \operatorname{Im} z \neq 0.$$

We also need the definition of the space $I'(r)$.

$I'(r)$ is the space of all $f \in S'$ for wich (a) holds and instead of (b), we suppose that there exist $C = C(F)$ and $\varepsilon = \varepsilon(F)$ such that

(3) $$|F(t)| \leq C(1 + |t|)^{r+m-\varepsilon}, t \in \mathbf{R}.$$

The Stieltjes transformation S_r of index r, $r \in \mathbf{R}\backslash(-\mathbf{N})$ of a distribution $f \in \mathcal{J}'(r)$ with the properties given in (2) is a complex valued function given by

(4)

$$\begin{aligned}(S_r f)(z) &= (r+1)_m \int_{-\infty}^{+\infty} F(t)(t+z)^{-r-m-1} dt \\ &= (r+1)_m < F(t), (t+z)^{-r-m-1} >, \text{ for } Im z \neq 0,\end{aligned}$$

where $(a)_n = a(a+1)\ldots(a+n-1)$, $n \in \mathbf{N}$ and $(a)_0 = 1$, $a \in \mathbf{R}$. It is easy to see that $S_r f$ is a holomorphic function of the complex variable z in the domain $\mathbf{C}\backslash(-\infty, +\infty)$. We shall use the following formulas from [3]. If $r > -1$, then

(5) $$(S_r \frac{H(t-1)}{t})(k) = (r+1) \int_1^{\infty} \frac{\ln t}{(k+t)^{r+2}} dt;$$

$$(6)\qquad (S_r\frac{H(t-1)}{t^s})(k)=\frac{1}{(s-1)!}[(s-2)!-\frac{1}{(1+k)^{r+1}}-$$

$$-\frac{(s-3)!(r+1)}{(1+k)^{r+2}}+\frac{(-1)^{s-2}(r+1)_{s-2}}{(1+k)^{r+s-1}}+\dots$$

$$+(-1)^{s-1}(r+1)_s\int_1^\infty\frac{\ln\ t}{(t+k)^{r+s+1}}dt],\ k>1,\ s\geq 2.$$

Let $f\in S'_+$ and

$$f(x)\overset{q\cdot e}{\sim}\sum_{i\in\Lambda}a_i\delta^{(i)}(x).$$

Then, because of $S_r(\delta^{(j)}(t))(x)=-\frac{(r+1)\dots(r+j)}{x^{r+j+1}}$, $r>-1, j\in\mathbf{N}_0$ we get

$$(S_rf)(x)\sim\sum_{i\in\Lambda}a_i\frac{(r+1)_i}{x^{r+i+1}},\ x\to\infty.$$

2. Application

Our intension in this paper is to give the application for the classical Stieltjes transformation. From the assertions which are to follow, one can easily derive the coresponding assertions for the distributional Stieltjes transformation. But we shall emphasize the fact that for the classical Stieltjes transformation we obtain new "classical" results by using the abstract theory of the preceding section.

We assume that $r\in\mathbf{N}_0$. First we shall prove two lemmas which will be used in the proof of the main Proposition 4.

Lemma 1. *Let $s\in\mathbf{N}, k>1$. Then*

$$(7)\qquad \int_1^\infty\frac{\ln t}{t(k+t)^{r+s+1}}dt=\frac{1}{k^{r+s}}\,(-\frac{1}{r+s}\ln k+$$

$$+\sum_{j=1}^\infty\frac{(-1)^{j-1}}{jk^j}\,\frac{\Gamma(r+s+j)}{\Gamma(j+1)\Gamma(r+s+1)}+\frac{1}{r+s}B_s)$$

where

$$B_s=\frac{(-1)^{r+s-1}}{\Gamma(r+s)}(\frac{1}{z}\ln\frac{z}{1+z})^{(r+s-1)}|_{z=1}+$$

$$+\int_1^\infty \frac{\ln u}{(1+u)^{r+s+1}}du.$$

Proof. We have

$$\int_1^\infty \frac{\ln t dt}{(t+k)^{r+s+1}} = \frac{1}{k^{r+s}}(\int_{1/k}^\infty \frac{\ln u}{(1+u)^{r+s+1}}du + \ln k \int_1^\infty \frac{du}{(1+u)^{r+s+1}} =$$

$$= \frac{1}{k^{r+s}}(\int_{1/k}^1 \frac{\ln u}{(1+u)^{r+s+1}}du + \int_1^\infty \frac{\ln u}{(1+u)^{r+s+1}}du + \frac{\ln k}{(r+s)} \frac{1}{(1+(1/k))^{r+s}})$$

and by the partial integration in the integral from $\frac{1}{k}$ to 1 we get

$$\int_1^\infty \frac{\ln t}{(t+k)^{r+s+1}}dt = \frac{1}{k^{r+s}}\left(\frac{1}{r+s}\int_{1/k}^1 \frac{du}{u(1+u)^{r+s}} + \int_1^\infty \frac{\ln u}{(1+u)^{r+s+1}}du\right).$$

Consider the function $F(z) = \int_{1/k}^1 \frac{du}{u(z+u)}$, $z > 0$. We have

$$F(z) = \frac{1}{z}(\ln u - \ln(z+u))\,|_{1/k}^1 = \frac{1}{z}(\ln(1+kz) - \ln(1+z)) =$$

$$= \frac{1}{z}(\ln k + \ln\frac{z}{1+z} - \ln(1+\frac{1}{kz})).$$

Take k such that $kz > 1$. Thus we get

$$F(z) = \frac{1}{z}\ln k + \frac{1}{z}\ln\frac{z}{1+z} + \sum_{j=1}^\infty \frac{(-1)^{j-1}}{jk^j z^{j+1}}.$$

After differentiation $(r+s-1)$ times we get

$$(F(z))^{(r+s-1)} = (-1)^{r+s-1}(r+s-1)!\int_{1/k}^1 \frac{du}{u(z+u)^{r+s}} =$$

$$= (-1)^{r+s}\frac{(r+s-1)!\ln k}{z^{r+s}} + (\frac{1}{z}\ln\frac{z}{1+z})^{(r+s-1)} +$$

$$+ \sum_{j=1}^\infty \frac{(-1)^{j-1}}{jk^j}\,\frac{\Gamma(j+r+s)(-1)^{r+s-1}}{\Gamma(j+1)z^{j+r+s}}$$

and thus by putting $z = 1$ we get

$$\int_{1/k}^1 \frac{du}{u(1+u)^{r+s}} = \ln k + \frac{(-1)^{r+s-1}}{\Gamma(r+s)}\left(\frac{1}{z}\ln\frac{z}{1+z}\right)^{(r+s-1)}\Big|_{z=1} +$$

$$+\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{jk^j}\frac{\Gamma(j+r+s)}{\Gamma(j+1)\Gamma(r+s)}$$

this implies (7).

We shall also the following formula

$$\frac{1}{(1+k)^s}=\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(s+j)}{\Gamma(s)\Gamma(j+1)}\frac{1}{k^{s+j}},\ k>1,\ s\in\mathbf{N}. \tag{8}$$

Let

$$T_1=\frac{a_2}{1}+\frac{a_3}{2}+\ldots+\frac{a_{n-1}}{n-2},\ T_2=\frac{a_3}{2!}+\frac{a_4}{3\cdot 2}+\ldots+\frac{a_{n-2}}{(n-2)(n-3)},$$

$$T_3=\frac{a_4}{3!}+\frac{a_5}{4\cdot3\cdot2}+\ldots+\frac{a_{n-1}(n-5)!}{(n-2)!},\ldots,T_{n-2}=\frac{a_{n-1}}{(n-2)!}.$$

Lemma 2. *Let* $a_i\neq 0, i\in\Lambda_{n-1}$

$$\sum_{i=1}^{n-1}(S_r\frac{H(t-1)}{t^i})(k)=\frac{1}{k^{r+1}}(a_1B_1+T_1)+\sum_{m=1}^{n-1}(-1)^{m-1}\frac{a_m(r+1)_m\ln k}{(m-1)!k^{r+m}(r+m)}+$$

$$+\sum_{\ell=2}^{n-2}(-1)^{\ell}\frac{\Gamma(r+\ell)}{k^{r+\ell}}(\sum_{i=1}^{\ell-1}\frac{a_i(r+1)_i}{(\ell-i)\Gamma(\ell+1-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{\ell-1}\frac{(r+1)_jT_{j+1}}{\Gamma(\ell-j)\Gamma(r+1+j)}+a_{\ell}\frac{(r+1)_{\ell}}{(\ell-1)!}B^*_{\ell})+$$

$$+\frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n-1}}(\sum_{i=1}^{n-2}\frac{a_i(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)}-\frac{a_{n-1}(r+1)_{n-1}B_{n-1}}{(n-2)!})+$$

$$+\frac{(-1)^n\Gamma(r+n)}{k^{r+n}}(\sum_{i=1}^{n-1}\frac{a_i(r+1)_i}{(n-i)\Gamma(n+1-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-j)\Gamma(r+1+j)})+0(\frac{1}{k^{r+n+1}}).$$

Proof. From formulas (5), (6), (7) and (8) we get ($k > 1$)

$$S_r \frac{H(t-1)}{t}(k) = \frac{1}{k^{r+1}}(\ln k + B_1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{jk^j} \frac{\Gamma(r+j+1)}{\Gamma(j+1)\Gamma(r+1)},$$

$$(S_r \frac{H(t-1)}{t^s})(k) = \frac{1}{(s-1)} \{(s-2)! \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(r+j+1)}{\Gamma(r+1)\Gamma(j+1)} \frac{1}{k^{r+1+j}} -$$

$$- (s-3)!(r+1) \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(r+j+2)}{\Gamma(r+2)\Gamma(j+1)} \frac{1}{k^{r+2+j}} + \dots$$

$$+(-1)^{s-2}(r+1)_{s-2} \sum_{j=2}^{\infty} \frac{(-1)^j \Gamma(r+s-1+j)}{\Gamma(j+1)\Gamma(r+s-1)} \frac{1}{k^{r+s-1+j}} +$$

$$+(-1)^{s-1}(r+1)_s \frac{1}{k^{r+s}}[\frac{1}{r+s} \ln k +$$

$$+\frac{1}{r+s} B_s + \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(r+s+j)}{jk^j \Gamma(j+1)\Gamma(r+s+1)}]\}$$

where $r = 2, 3, \dots, n-1$. We shall denote by $B_s^* = \frac{1}{r+s} B_s$.

This implies

$$H(k) = \sum_{i=1}^{n-1} a_i (S_r \frac{H(t-1)}{t^i})(k) =$$

$$= a_1[\frac{\ln k}{k^{r+1}} + \frac{B_1}{k^{r+1}} + \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(r+j+1)}{\Gamma(j+1)\Gamma(r+1) j k^{r+j+1}}] +$$

$$+ a_2[\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(r+j+1)}{\Gamma(r+1)\Gamma(j+1)} \frac{1}{k^{r+1+j}} - \frac{(r+1)_2}{k^{r+2}} \frac{\ln k}{r+2} - \frac{(r+1)_2}{k^{r+2}} B_2^* -$$

$$-(r+1)_2 \sum_{j=1}^{\infty} \frac{(-1)^{j-1} \Gamma(r+j+2)}{jk^{j+r+2} \Gamma(j+1)\Gamma(r+3)}] +$$

....

$$+\frac{a_s}{(s-1)!}[\sum_{i=0}^{s-2}(-1)^i (s-2-i)!(r+1)_i \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(r+1+j+i)}{\Gamma(j+1)\Gamma(r+1+i)} \frac{1}{k^{r+1+i+j}} +$$

$$+ \frac{(-1)^{s-1}(r+1)_s \ln k}{k^{r+s}(r+s)} + (-1)^{s-1}(r+1)_s \frac{B_s^*}{k^{r+s}} +$$

$$+(-1)^{s-1}(r+1)_s\sum_{j=1}^{\infty}\frac{(-1)^{j-1}\Gamma(r+s+j)}{jk^{r+s+j}\Gamma(j+1)\Gamma(r+s+1)}+$$

$$+\frac{a_{n-1}}{(n-2)!}\,[\sum_{i=0}^{n-3}(-1)^i(n-3-i)!(r+1)_i\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(r+1+j+i)}{\Gamma(j+1)\Gamma(r+1+i)}\,\frac{1}{k^{r+1+i+j}}+$$

$$+(-1)^{n-2}(r+1)_{n-1}\frac{\ln k}{k^{r+n-1}(r+n-1)}+(-1)^{n-1}\frac{(r+1)_{n-1}}{k^{r+n-1}}B^*_{n-1}+$$

$$+(-1)^{n-2}(r+1)_{n-1}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}\Gamma(r+n+1+j)}{jk^{r+n-1+j}\Gamma(j+1)\Gamma(r+n)}\;=$$

$$=\frac{1}{k^{r+1}}\,[a_1B_1+(a_2+\frac{a_3}{2}+\ldots+\frac{a_{n-1}}{n-2})\,]+a_1\frac{\ln k}{k^{r+1}}+\frac{1}{k^{r+2}}\,(-a_1\frac{\Gamma(r+2)}{\Gamma(2)\Gamma(r+1)}-$$

$$-a_2(r+1)_2B^*_2-a_2\frac{\Gamma(r+2)}{\Gamma(r+1)\Gamma(2)}-a_3\frac{\Gamma(r+2)}{\Gamma(r+1)\Gamma(2)}-\frac{a_3(r+1)}{2}\,\frac{\Gamma(r+2)}{\Gamma(1)\Gamma(r+2)}-$$

$$-\frac{a_4}{3}\,\frac{\Gamma(r+2)}{\Gamma(r+1)\Gamma(2)}-\frac{a_4}{3!}(r+1)\frac{\Gamma(r+2)}{\Gamma(1)\Gamma(r+2)}-\frac{a_{n-1}}{(n-2)}\,\frac{\Gamma(r+1)}{\Gamma(1)\Gamma(r+1)}-$$

$$-\frac{a_{n-1}}{(n-2)!}(n-4)!\,\frac{(r+1)\Gamma(r+2)}{\Gamma(r+2)\Gamma(1)}\,)-a_2(r+1)_2\frac{\ln k}{k^{r+2}}-(r+1)a_2\frac{B_2}{k^{r+2}}+\ldots+$$

$$+\frac{(-1)^{s-2}}{k^{r+s}}\,[\frac{(r+1)\Gamma(r+s)}{\Gamma(s)\Gamma(r+1)(s-1)}a_1+\frac{(r+1)_2\Gamma(r+s)a_2}{\Gamma(s-1)\Gamma(r+2)(s-2)}+\ldots+$$

$$+\frac{(r+1)_{s-1}a_{s-1}}{(s-2)!}\,\frac{\Gamma(r+s)}{\Gamma(2)\Gamma(r+s-1)})+\frac{\Gamma(r+s)}{\Gamma(s)\Gamma(r+1)}T_1+$$

$$+\frac{(r+1)\Gamma(r+s)}{\Gamma(s-1)\Gamma(r+2)}\,T_2+\ldots+\frac{\Gamma(r+s)(r+1)_{\ell-1}}{\Gamma(1)\Gamma(r+s)}\,T_s+\frac{a_s(r+1)_s}{(s-1)!}\,B^*_s\,]+\ldots+$$

$$+\frac{(-1)^{n-2}(r+1)_{n-1}a_{n-1}}{(n-2)!(r+n-1)k^{r+n-1}}\,\ln k+\ldots$$

After some calculation we get

$$H(k)=\frac{1}{k^{r+1}}(a_1B_1+T_1)+a_1\frac{\ln k}{k^{r+1}}-\frac{1}{k^{r+2}}\,[\frac{\Gamma(r+2)}{\Gamma(2)\Gamma(r+1)}a_1+\frac{\Gamma(r+2)}{\Gamma(2)\Gamma(r+1)}T_1+$$

$$+\frac{\Gamma(r+2)(r+1)T_2}{\Gamma(1)\Gamma(r+2)}-a_2(r+1)_2B^*_2\,]-a_2(r+1)_2\frac{\ln k}{k^{r+2}}+$$

$$+\frac{\Gamma(r+3)}{k^{r+3}}\,[-\sum_{i=1}^{2}\frac{a_i(r+1)_i}{(3-i)\Gamma(4-i)\Gamma(r+i)(i-1)!}+\sum_{i=0}^{2}\frac{(r+1)_jT_{j+1}}{\Gamma(3-j)\Gamma(r+1+j)}+$$

$$+\frac{a_3(r+1)_3}{2}B_3^*\,]+\dots$$

$$\dots+\frac{(-1)^{n-2}\Gamma(r+n-2)}{k^{r+n-2}}\,[\sum_{i=1}^{n-3}\frac{a_i(r+1)_i}{(n-2-i)\Gamma(n-1-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{i=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-2-j)\Gamma(r+1+j)}-\frac{a_{n-2}(r+1)_{n-2}B_{n-2}^*}{(n-3)!}\,]+$$

$$+(-1)^{n-2}\frac{(r+1)_{n-1}\ln k\; a_{n-1}}{(n-2)!(r+n-1)k^{r+n-1}}+$$

$$+(-1)^{n-1}\frac{\Gamma(r+n-1)}{k^{r+n-1}}\,(\sum_{i=0}^{n-2}\frac{a_i(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)}-\frac{a_{n-1}(r+1)_{n-1}}{(n-2)!}B_{n-1}^*\,)+\dots$$

This proves the assertion.

For the next proposition we also need the following assertion from [3].

Lemma 3. *If* $h\in L^1_{loc}\,(0,\infty)$ *and* $h(t)\sim\frac{a_n}{t^n}$, $t\to\infty$. *Then*

$$(S_rh)(k)=a_n\,(\frac{m_0}{k^{r+1}}-\frac{(r+1)m_1}{k^{r+2}}+\dots+(-1)^{n-2}\frac{(r+1)_{n-2}m_{n-2}}{k^{r+n-1}}+$$

$$+\frac{(-1)^{n-2}}{(n-1)!}\,\frac{(r+1)_n}{r+n}\,\frac{\ln k}{k^{r+n}}\,)+0\,(\frac{\ln k}{k^{r+n}}\,),\ k\to\infty,$$

where $m_i=\int_0^\infty t^i\frac{h(t)}{i!}dt$, $i=0,\dots,n-2$.
Now, from the previous two Lemmas we obtain

Proposition 1. *Let* $f\in L^1_{loc}\,(0,\infty)$ *and let* $f(t)\sim\sum_{i=1}^{\infty}\frac{a_i}{t^i}$, $t\to\infty$. *Then for* $n\geq 2$ *we have*

$$(S_rf(t))(k)\sim\frac{1}{k^{r+1}}(a_1B_1+T_1+a_nm_0)+\sum_{m=1}^{n-1}\frac{(-1)^m a_m(r+1)_{m-1}\ln k}{(m-1)!k^{r+m}}+$$

$$+\sum_{\ell=2}^{n-2}\frac{(-1)^\ell}{k^{r+\ell}}(\Gamma(r+\ell)\sum_{i=1}^{\ell-1}\frac{a_i(r+1)_i}{(\ell-1)\Gamma(\ell+1-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{\ell-1}\frac{(r+1)_jT_{j+1}}{\Gamma(\ell-j)\Gamma(r+1+j)}+\frac{a_\ell(r+1)_\ell B^*_\ell}{(1-1)!}-m_{\ell-1}(r+1)_{\ell-1})+$$

$$+\frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n-1}}(\sum_{i=1}^{n-2}\frac{a_i(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)}-\frac{a_{n-1}(r+1)_{n-1}B^*_{n-1}}{(n-1)!}-m_{n-1}(r+1)_{n-2})+$$

$$+\frac{(-1)^{n-1}}{(n-1)!}(r+1)_{n-1}\frac{\ln k}{k^{r+n}}+o(\frac{\ln k}{k^{r+n}}),\, as\, k\to\infty$$

where

$$m_i=\int_0^\infty t^i\,\frac{f(t)-\sum_{j=1}^{n-1}\frac{a_jH(t-1)}{t^j}}{i!}dt,\quad i=1,\ldots,n-2.$$

Proof. From the assumptions we have

$$f(t)-\sum_{i=1}^{n-1}a_i\frac{H(t-1)}{t^i}\sim\frac{a_n}{t^n},\ t\to\infty.$$

By Lemma 3 we get

$$(S_rf(t))(k)-\sum_{i=1}^{n-1}a_i(S_r\frac{H(t-1)}{t^i})(k)\ =$$

$$=\ a_n\sum_{\ell=0}^{n-2}(-1)^\ell\frac{m_\ell(r+1)_\ell}{k^{r+\ell+1}}+\frac{(-1)^{n-1}}{(n-1)!}\,\frac{(r+1)_{n-1}}{k^{r+n}}\ln k+o(\frac{\ln k}{k^{n+r}})$$

Now, by Lemma 2, we have

$$(S_rf(t))(k)=\frac{1}{k^{r+1}}(a_1B_1+T_1)+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}a_m(r+1)_m\ln k}{(m-1)!k^{r+m}}+$$

$$+\sum_{\ell=2}^{n-2}\frac{(-1)^\ell\Gamma(r+\ell)}{k^{r+\ell}}(\sum_{i=1}^{\ell}\frac{a_i(r+1)_i}{((\ell-i)\Gamma(\ell+1-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{\ell-1}\frac{(r+1)_jT_{j+1}}{\Gamma(\ell-j)\Gamma(r+j)}+\frac{a_\ell(r+1)_\ell B_\ell^*}{(n-2)!})+$$

$$\frac{(-1)^{n-1}\Gamma(r+n-1)}{k^{r+n-1}}(\sum_{i=1}^{n-1}\frac{a_i(r+1)_i}{(n-1-i)\Gamma(n-i)\Gamma(r+i)(i-1)!}-$$

$$-\sum_{j=0}^{n-3}\frac{(r+1)_jT_{j+1}}{\Gamma(n-1-j)\Gamma(r+j)}-\frac{a_{n-1}(r+1)_{n-1}B_{n-1}^*}{(n-2)!})+o(\frac{\ln k}{k^{r+n}})+$$

$$+a_n\sum_{\ell=0}^{n-2}(-1)^\ell\frac{m_\ell(r+1)_\ell}{k^{r+\ell+1}}+\frac{(-1)^{n-1}}{(n-1)!}\frac{(r+1)_{n-1}}{k^{r+n}}\ln k.$$

This implies the assertion.

Clearly the cases when r is not form N_0 and when instead of $\sum_{i=0}^{\infty}\frac{a_i}{t^i}$ we have $\sum_{i=1}^{\infty}\frac{a_iL_i(t)}{t^i}$ in the assumptions of Proposition 4, then the corresponding assertion can be derived in a similar but technically more complicated way.

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REZIME

KVAZIASIMPTOTSKI RAZVOJ I STIELTJESOVA TRANSFORMACIJA

Koristeći pojam kvaziasimptotskog razvoja temperirane distribucije u ∞ dobijamo, Tvrdjenje 4, (običan) asimptotski razvoj Stieltjesove transformacije funkcije $f \in L^1_{loc}(0,\infty)$ koja ima asimptotski razvoj oblika $\sum_{i=1}^{\infty} \frac{a_i}{t^i}$, $t \to \infty$.

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