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REPRESENTATION FOR A CLASS OF NON – LINEAR FUNCTIONALS OF GAUSSIAN MARTINGALES

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Abstract

Let $\{X(t), t > 0\}$ be a continuous Gaussian martingale and let $\mathcal{H}^*$ be the mean-square linear closure of all the one-dimensional polynomials $\{P_n(X(t)), n = \overline{1,\infty}, t > 0\}$. For $Y \in \mathcal{H}^*$, there is the representation $Y = \int_0^\infty \Phi(t, X(t))\, dX(t)$, $\|\Phi(t, X(t))\| \in \mathcal{L}_2(\|dX(t)\|^2)$.

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1. Introduction

Let $\{X(t), t > 0\}, X(0) = 0$, be a real mean-square continuous martingale and let the Hilbert space $\mathcal{H}_1$ be the mean-square linear closure of $\{X(t), t > 0\}$. It is a well-known fact (see, for instance, [2], Ch. IX) that any $Y, Y \in \mathcal{H}_1$ has the representation

$$Y = \int_0^\infty \varphi(t) dX(t), \tag{1}$$

where the non-random function $\varphi(u), u > 0$, belongs to $\mathcal{L}_2(dF), F(t) = EX^2(t) = \|X(t)\|^2$. In this paper we shall consider the continuous Gaussian martingale $\{X(t), t > 0\}$ and the Hilbert space $\mathcal{H}^*$ -the mean-square linear closure of all the one-dimensional polynomials $\{P_n(X(t)), n = \overline{1,\infty}, t > 0\}$. (We assume that all the random variables are centered at the expectations). Remark that $\mathcal{H}^*$ is the subspace of $\mathcal{H}$ -the linear closure of all the polynomials $\{P_n(X(t_1)\dots,X(t_n)), n = \overline{1,\infty}, t_1,\dots,t_n > 0\}$. For example, the element $X(t)X(s) - F(min\{s,t\}), s \neq t$, does not belong to $\mathcal{H}^*$. We shall show that $Y \in \mathcal{H}^*$ has the representation

$$Y = \int_0^\infty \Phi(t, X(t))dX(t), \tag{2}$$

where $\|\Phi(t, X(t))\|$ belongs to $\mathcal{L}_2(dF)$. The integral of form (2) is considered in the sense of [2], Ch.IX.

The proof proceeds according to the technique of Hermite polynomials $H_p(X_1,\dots,X_p)$ of Gaussian variables $X_1,\dots,X_p$ (see, for instance, [3]). We write $H_p(X) = H_p\underbrace{(X_1,\dots,X)}_{k}$

Lemma 1. *For* $0 < t < t_1, \Delta X = X(t_1) - X(t), p \geq 2$

$$[H_p(X(t) + \Delta X) - H_p(X(t)] - pH_{p-1}(X(t))\Delta X = \tag{3}$$

$$= A_2(\Delta X)^2 + \dots + A_p(\Delta X))^p,$$

where A_k *are independent of* ΔX.

Proof. Recall the relation $H_p(X+Y) = \sum_{k=0}^p \binom{p}{k} H_p\underbrace{(X,\dots,X}_{k}\underbrace{Y,\dots,Y)}_{p-k}$, [4], and the factorization property of Hermite polynomials: if X and Y are independent, then $H_p\underbrace{(X,\dots,X}_{k}\underbrace{Y,\dots,Y)}_{p-k} = H_k(X)H_{p-k}(Y)$. Since $X(t)$ and ΔX are independent, we have

$$H_p(X(t) + \Delta X) = H_p(X(t)) + pH_{p-1}(X(t))H_1(\Delta X) +$$

$$+ \binom{p}{p-2} H_{p-2}(X(t))H_2(\Delta X) + \dots$$

$$\dots + pH_1(X(t))H_{p-1}(\Delta X) + H_p(\Delta X). \qquad \square$$

2. Representation

Proposition. 1. *The space $\mathcal{H}^*$ coincides with the set of integrals $\int_0^\infty \Phi(t,X(t))$ $dX(t)$, where the random function $\Phi(t,X(t))$ satisfies $\|\Phi(t,X(t))\| \in \mathcal{L}_2(dF)$ (It is not necessarily that $E\Phi = 0$).*

Proof. Let $\mathcal{H}_p^*, p = \overline{1,\infty}$, be the mean-square linear closure of $\{H_p(X(t)), t > 0\} (\mathcal{H}_1^* = \mathcal{H}_1)$. Then, by the orthogonality of Hermite polynomials of different degrees, it holds that

$$\mathcal{H}^* = \sum_{p=1}^{\infty} \bigotimes \mathcal{H}_p^* \tag{4}$$

Let E_t be the conditional expectation with respect to δ-field generated by $\{X(u),\ u \leq t\}$. From

$$E_s H_p(X(t)) = H_p(E_s X(t)) = H_p(X(s)), s < t, \text{ [3]} \tag{5}$$

it follows that $H_p(X(t)), t > 0\}$ is a martingale. In this way $\mathcal{H}_p^*$ coincides with the set of integrals

$$\int_0^\infty \varphi_p(t) dH_p(X(t)), \varphi_p(t) \in \mathcal{L}_2(\|dH_p(X(t))\|^2). \tag{6}$$

By Lemma $dH_p(X(t)) = pH_{p-1}(X(t))dX(t), H_0(\cdot) = 1.$ Then (6) becomes

$$\int_0^\infty \varphi_p(t) p H_{p-1}(X(t)) dX(t). \tag{7}$$

Remark that the measure $\|dH_p(X(t))\|^2 = \|pH_{p-1}(X(t))\|^2 dF(t)$ is equivalent by absolute continuity to the measure $dF(t)$. Let $Y \in \mathcal{H}^*$. By (4) and (6)

$$Y = \sum_{p=1}^{\infty} \int_0^\infty \varphi_p(t) p H_{p-1}(X(t)) dX(t) = \int_0^\infty \{\sum_{p=1}^{\infty} \varphi_p(t) p H_{p-1}(X(t))\} dX(t).$$

Let us denote $\Phi(t,X(t)) = \sum_{p=1}^{\infty} \varphi_p(t) p H_{p-1}(X(t))$. The random function $\Phi(t,X(t)), t > 0$, is such that $\|\Phi(t,X(t)\| \in \mathcal{L}_2(dF)$.
Conversely, consider the random function $\Phi(t,X(t)), t > 0$, such that $\int_0^\infty \|\Phi(t,X\ (t))\|^2 dF(t) < \infty$.
For almost every t (with respect to dF), $\|\Phi(t,X(t))\|^2$ is finite. The random variable $\Phi(t,X(t)), t \in \sup dF$ is decomposable by the complete

orthogonal system $\{H_p(X(t)), p = \overline{0,\infty}\}$ in the space of all the random variables of the finite variances and measurable to $X(t)$. Let $\Phi(t, X(t)) = \sum_{p=0}^{\infty} \varphi_p(t) H_p(X(t))$ be this decomposition.
The integral

$$Y = \int_0^\infty \Phi(t, X(t)) dX(t) = \int_0^\infty \{\sum_{p=0}^{\infty} \varphi_p(t) H_p(X(t))\} dX(t) =$$

$$= \sum_{p=0}^{\infty} \int_0^\infty \varphi_p(t) \frac{1}{p+1} dH_{p+1}(X(t))$$

belonges to $\mathcal{H}^*$. □

3. Processes as curves in $\mathcal{H}^*$. Linear completeness.

Let us denote $\mathcal{H}_t^* = E_t \mathcal{H}^*$. By (5) the subspace $\mathcal{H}_t^*$ is the linear closure of $\{P_n(X(u)), n = \overline{1,\infty}, u \leq t\}$ and $\mathcal{H}_p^*$ reduces the family of projections $\{E_t, t > 0\}$. It follows immediately that the spectral type of $\{E_t, t > 0\}$ in $\mathcal{H}^*$ is dF and the multiplicity of dF is ∞. (For these notions see, for instance, the classical paper [1]). Consider the process $\{Y(t), t > 0\}$in æ* defined by

$$Y(t) = \int_0^t \Phi(t, u, X(u)) dX(u), \|\Phi(t, \cdot, X(\cdot)\| \in \mathcal{L}_2(dF).$$

Let us benote by $\mathcal{H}_1(Y;t)$ then mean-square linear closure of $\{Y(u), u \leq t\}$. We say that the process $\{Y(t)\}$ is linearly complete in $\mathcal{H}^*$ if $\mathcal{H}_1(Y;t) = \mathcal{H}_t^*$ for each $t > 0$. The characterization of the completenes of $\{Y(t)\}$ in the terms of $\Phi(t, u, X(u))$ seems to be of some interest. We say that the family of random functions $\{\Phi(t, u, X(u)), t > 0\}$ is complete in $\mathcal{H}^*$ if from $\int_0^t < \Phi(t, u, X(u), \Psi(u, X(u)) > dF(u) = 0$, for each $t > 0$, follows that $\in t_0^\infty \|\Psi(u, X(u))\|^2 dF(u) = 0.$ ($< \cdot, \cdot >$ denotes the inner product in $\mathcal{H}^*$.

Proposition. 2. *The relation $\mathcal{H}_1(Y;t) = \mathcal{H}_t^*$ for each $t > 0$ holds if and only if the family $\{\Phi(t, u, X(u)), t > 0\}$ is complete in $\mathcal{H}^*$.*

Proof. Let $Z = \int_0^\infty \Psi(u, X(u)) dX(u) \in \mathcal{H}^*$. Then $< Y(t), Z >= \int_0^t < \Phi(t, u, X(u), \Psi(u, X(u)) > dF(u)$. If $< Y(t), Z >= 0$ for each $t > 0$, then Z

is orthogonal to $\mathcal{H}_1(Y)$. So, if $Z \in \mathcal{H}_1(Y;t)$ it follows that $\|Z\|^2 = \int_0^\infty \|\Psi(u,X(u))\|^2 dF(u) = 0$ or the family $\{\Phi(t;u,X(u)),t > 0\}$ is complete in $\mathcal{H}^*$. Conversely, if $\{\Phi(t,u,X(u)),t > 0\}$ is complete, it follows that necessarily $Z = 0$ or $\mathcal{H}_1(Y;t) = \mathcal{H}_t^*$. Remark that the completeness of $\{\Phi(t,u,X(u)),t > 0\}$ in $\mathcal{H}^*$ is equivalent to the completeness of the system of non-random functions $\{\varphi_p(t,n),t > 0,p = \overline{1,\infty}\}in\mathcal{H}^*$, because $\Phi(t,u,X(u)) = \sum_{p=1}^\infty \varphi_p(t,u)pH_{p-1}(X\ (u))$.

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REZIME

REPREZENTACIJA JEDNE KLASE NELINEARNIH FUNKCIONALA GAUSOVSKIH MARTINGALA

Neka je $\{X(t),t > 0\}$ neprekidni Gausovski martingal i neka je $\mathcal{H}^*$ srednje kvadratna linearna zatvorenost svih jedno-dimenzionalnih polinoma $\{P_n(X(t)),n = \overline{1,\infty},t > 0\}$. Za $Y \in \mathcal{H}^*$ važi reprezentacija $Y = \int_0^\infty \Phi(t,X(t))dX(t), \|\Phi(t,X(t))\| \in \mathcal{L}_2(\|dX(t)\|^2)$.

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