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A NOTE ON CONFORMALLY QUASI – RECURRENT MANIFOLDS

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Abstract

In §1 the condition for the conformally quasi-recurrent manifold (CQR-manifold for short) is found to be conformally recurrent.In §2 the conformal change of CQR- manifold is disscused and some results of [6] completed. In §3 some examples of CQR-manifolds are given. In §4 the umbilical hypersurface of the conformally quasi-recurrent space is studied.

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1. Introduction

Let (M,g) be an n-dimensional$(n > 3)$ Riemannian space and let C^h_{ijk} be its conformal curvature tensor, i.e.

$$(1.1) \qquad C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(R^h_k g_{ij} - R^h_j g_{ik} + \delta^h_k R_{ij} - \delta^h_j R_{ik}) +$$

$$+\frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

where R^h_{ijk} denotes the curvature tensor of (M,g), R_{ij} is the Ricci tensor, while R is the scalar curvature. It is well known that tensor (1.1) satisfies the relations

$$C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}, \tag{1.2}$$

$$C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0, \tag{1.3}$$

$$C^p_{pjk} = C^p_{jpk} = C^p_{jkp} = 0. \tag{1.4}$$

(M,g) is said to be conformally recurrent (CR for short), if C^h_{ijk} satisfies the condition

$$\nabla_s C_{hijk} = a_s C_{hijk}, \tag{1.5}$$

where ∇ is the operator of the covariant derivative with respect to metric g and a_s is a vector field.

(M,g) is said to be conformally quasi-recurrent (CQR for short) if

$$\nabla_s C_{hijk} = 2a_s C_{hijk} + a_h C_{sijk} + a_i C_{hsjk} + a_j C_{hisk} + a_k C_{hijs}. \tag{1.6}$$

If in (1.5) or in (1.6) $a_i = 0$, we have

$$\nabla_s C_{hijk} = 0. \tag{1.7}$$

The Riemannian space satisfyng (1.7) is said to be conformally symmetric (CS for short).

Thus, the class of CR-manifolds as well as the class of CQR-manifolds contains all CS-manifolds. There arises the question: do there exist manifolds which are not CS-manifolds but are CR - as well as CQR-manifolds? And if such manifolds exist, do there exist CQR-manifolds which are not CR-manifolds and CR-manifolds which are not CQR-manifolds?

To answer these questions, we shall first mention the following

Lemma 1. *([7], Lemma 3) If c_j, p_j and B_{hijk} are numbers satisfying*

$$c_s B_{hijk} + p_h B_{sijk} + p_i B_{hsjk} + p_j B_{hisk} + p_k B_{hijs} = 0,$$

$$B_{hijk} = B_{jkhi} = -B_{hikj}, \quad B_{hijk} + B_{hjki} + B_{hkij} = 0,$$

then each $b_j = c_j + 2p_j$ is zero or each B_{hijk} is zero.

Now, we can easily prove

Theorem 1. *The necessary and sufficient condition for a CR-manifold to be a CQR-manifold and for a CQR-manifold be a CR-manifold is*

$$a_s C_{hijk} + a_j C_{hiks} + a_k C_{hisj} = 0. \tag{1.8}$$

Proof. - Let (M, g) be CR and CQR-manifold. Then, besides (1.6), the relation

$$\nabla_s C_{hijk} = k_s C_{hijk}$$

is satisfied, too. Therefore, we have

$$(2a_s - k_s) C_{hijk} + a_h C_{sijk} + a_i C_{hsjk} + a_j C_{hisk} + a_k C_{hijs} = 0. \tag{1.9}$$

In view of (1.2) and (1.3), the conditions of the preceding Lemma are satisfied. Supposing that (M, g) is not conformally flat, it follows that $k_s = 4a_s$, Hence, (1.9) takes the form

$$-2a_s C_{hijk} + a_h C_{sijk} + a_i C_{hsjk} + a_j C_{hisk} + a_k C_{hijs} = 0. \tag{1.10}$$

Permuting the indices s, j, k, cyclically, adding the obtained equations to the preceding one and using (1.2) and (1.3), we obtain (1.8). Now, let us suppose that (1.8) holds good. Then, taking into account (1.2), we also have.

$$a_s C_{hijk} + a_h C_{isjk} + a_i C_{shjk} = 0$$

Adding this to (1.8), we get (1.10). Therefore, (1.6) reduces to

$$C_{hijk} = 4a_s C_{hijk} \tag{1.11}$$

and (1.11) reduces to (1.6). This completes the proof of theorem.

As an immediate consequence of (1.8) and (1.4), we have

$$a_s C^s_{ijk} = 0. \tag{1.12}$$

The converse, of course, is not true, i.e. (1.8) does not follow from (1.12).

On the other hand, relation (1.12) is satisfied for any CQR-manifold [6]. Thus: a CQR-manifold which is not a CR-manifold is that satisfyng (1.6) (and consequently (1.12)), but not satisfyng (1.8). In §3 we give examples of such manifolds.

A CR-manifold which is not a CQR-manifold is that which does not satisfy (1.8). Example 2 given in [9] is an example of such a manifolds.

(M, g) which is a CR-manifold as well as a CQR-manifold is that which satisfies (1.5) (or (1.6)) and (1.8). We call such a manifold a CRQR (conformally recurrent and quasi-recurrent) manifold. In §2 we mention examles of CRQR-manifolds.

We have shown in [6] that if the vector field a_i in (1.6) is a gradient vector field, a CQR-manifold can be conformally related to a CS-one. In this paper, in §2, we discuss the conformal change of whichever CQR-manifold. Particularly, we investigate the conformal change of a CQR-manifold into a CQR-manifold as well as into a CR-one.

In §4 we study the umbilical hypersurface of a conformally quasi-recurrent manifold.

2. Conformal change of a CQR-manifold

Let us consider the conformal change

$$\tilde{g}_{ij} = e^{2\sigma} g_{ij}. \tag{2.1}$$

The Christoffel symbols of metrics $\tilde{g}$ and g are related as follows

$$\left\{ \begin{matrix} \tilde{h} \\ ij \end{matrix} \right\} = \left\{ \begin{matrix} h \\ ij \end{matrix} \right\} + \delta_i^h \sigma_j + \delta_j^h \sigma_i - g_{ij}\sigma^h, \quad \sigma_i = \frac{\partial \sigma}{\partial x^i},$$

while the conformal curvature tensor is invariant

$$\tilde{C}^h_{ijk} = C^h_{ijk}. \tag{2.2}$$

Let $\tilde{\nabla}$ be the operator of the covariant derivative with respect to metric $\tilde{g}$. Applying it to (2.2), we get

$$\tilde{\nabla}_s \tilde{C}^h_{ijk} = \nabla_s C^h_{ijk} \tag{2.3}$$

$$- \quad 2\sigma_s C^h_{ijk} - \sigma^h C_{sijk} - \sigma_i C^h_{skj} - \sigma_j C^h_{isk} - \sigma_k C^h_{ijs} +$$
$$+ \quad \delta^h_s \sigma_r C^r_{ijk} + g_{is}\sigma^r C^h_{rjk} + g_{js}\sigma^r C^h_{irk} + g_{ks}\sigma^r C^h_{ijr}.$$

Now, let us suppose that both metrics g and $\tilde{g}$ are conformally quasi-recurrent. This means that beside (1.6) and (1.12), we have

$$\tilde{\nabla}_s \tilde{C}^h_{ijk} = 2b_s \tilde{C}^h_{ijk} + \tag{2.4}$$

$$+ b^h \tilde{C}_{sijk} + b_i \tilde{C}^h_{sjk} + b_j \tilde{C}^h_{isk} + b_k \tilde{C}^h_{ijs},$$

$$b_s \tilde{C}^s_{ijk} = 0. \tag{2.5}$$

Substituting (1.6) and (2.4) into (2.3) and taking into account (2.2), we find

$$2(b_s - a_s + \sigma_s)C^h_{ijk} + (b^h - a^h + \sigma^h)C_{sijk} + (b_i - a_i + \sigma_i)C^h_{sjk} + \tag{2.6}$$

$$+ (b_j - a_j + \sigma_j)C^h_{isk} + (b_k - a_k + \sigma_k)C^h_{ijs} =$$

$$= \delta^h_s \sigma_r C^r_{ijk} + g_{is}\sigma^r C^h_{rjk} + g_{js}\sigma^r C^h_{irk} + g_{ks}\sigma^r C^h_{ijr}.$$

Transvecting with respect to h and s and using (1.12), (2.5) and (2.2), we get

$$(n - s)\sigma_r C^r_{ijk} = 0$$

or

$$\sigma_r C^r_{ijk} = 0, \tag{2.7}$$

because of $n > 3$. Therefore, (2.6) reduces to

$$2(b_s - a_s + \sigma_s)C^h_{ijk} + (b^h - a^h + \sigma^h)C_{sijk} + (b_i - a_i + \sigma_i)C^h_{sjk} +$$

$$+(b_j - a_j + \sigma_j)C^h_{isk} + (b_k - a_k + \sigma_k)C^h_{ijs} = 0,$$

from wich, according to the Lemma of §1, there follows $b_s - a_s + \sigma_s = 0$. Thus, we have

Theorem 2. *The conformal change (2.1) transforms a CQR-manifold into a CQR-manifold if and only if function σ satisfies condition (2.7). The corresponding vector fields are related as follows*

$$b_i = a_i - \sigma_i.$$

If in (1.6) vector field a_i is a gradient vector field, then we can choose function σ in (2.1) such that $a_i = \sigma_i$, condition (2.7) being satisfied because of (1.12). But, then $b_i = 0$, and (M, $\tilde{g}$) is a CS-manifold. So,

Corollary 1. [6].- *A CQR-manifold (1.6) whose vector field a_i is a gradient vector field, can be conformally related to a CS-manifold.*

Roter proved in [7] facts which we can summarize as follows:

The conformal change (2.1) transforms a CR-manifold into a CR-manifold if and only if function σ satisfies the condition

$$\sigma_s C_{hijk} + \sigma_j C_{hiks} + \sigma_k C_{hisj} = 0. \tag{2.8}$$

The corresponding vector fields are related as follows

$$b_i = a_i - 4\sigma_i.$$

If in (1.5), vector field a_i is a gradient vector field satisfyng (1.8), we can choose function σ in (2.1) such that $\sigma_i = \frac{a_i}{4}$. Then $b_i = 0$ and (M,$\tilde{g}$) is a CS-manifold. Thus, we have

Corollary 2. *A CR-manifold (1.5) whose vector field a_i is a gradient and satisfies (1.8), can be conformally related to a CS-manifold.*

The CR-manifolds of Corollary 2 are called by Roter [8] SCR (special conformally recurrent) manifolds. According to the results obtained in §1, we have

Corollary 3. *Any SCR-manifold is a CRQR-manifold.*

Looking at Corollaries 1 and 2 on the part of a CS-manifold, we can say as follows:

Theorem 3. *If a CS-manifold allows function σ satisfyng (2.8), it can be conformally related to a CRQR-manifold.*

If a CS-manifold allows function σ satisfyng (2.7) but not satisfyng (2.8), it can be conformally related to an essentially CQR-manifold (that is, to a CQR-manifold which is not a CRQR-manifold).

So, to construct examples of essentially CQR-manifolds, it is sufficient to find a CS-manifold allowing function σ, such that the conditions of the second part of Theorem 3 are satisfied. We shall show in §3 that such SC-manifolds exist.

In the remaining part of this §, we shall investigate the conformal change of a CQR-manifold into a CR-manifold. In other words, we shall suppose that metric g satisfies (1.6), while metric $\tilde{g}$ satisfies

$$\tilde{\nabla}_s \tilde{C}^h_{ijk} = c_s \tilde{C}^h_{ijk}. \tag{2.9}$$

Substituting (1.6) and (2.9) into (2.3) and taking into account (2.2), we find

$$c_s C^h_{ijk} = 2(a_s - \sigma_s)C^h_{ijk} + (a^h - \sigma^h)C_{sijk} + (a_i - \sigma_i)C^h_{sjk} + \tag{2.10}$$

$$+ (a_j - \sigma_j)C^h_{isk} + (a_k - \sigma_k)C^h_{ijs} +$$

$$+ \delta^h_s \sigma_r C^r_{ijk} + g_{is}\sigma^r C^h_{rjk} + g_{js}\sigma^r C^h_{irk} + g_{ks}C^h_{ijr}\sigma^r.$$

Now, following step by step the way described in [7], we can conclude:

If the conformal change (2.1) transforms a CQR-manifold into a CR-manifold, function σ satisfies condition (2.7).

Substituting (2.7) into (2.10), we get

$$(2a_s - c_s - 2\sigma_s)C^h_{ijk} =$$

$$= (a^h - \sigma^h)C_{sijk} + (a_i - \sigma_i)C^h_{sjk} + (a_j - \sigma_j)C^h_{isk} + (a_k - \sigma_k)C^h_{ijs} = 0,$$

from which follows, according to the Lemma of §1,

$$c_s = 4(a_s - \sigma_s). \tag{2.11}$$

Therefore, the preceding relation reduces to

$$-2(a_s - \sigma_s)C_{hijk} + (a_h - \sigma_h)C_{sijk} + (a_i - \sigma_i)C_{hsjk} + \\ + (a_j - \sigma_j)C_{hisk} + (a_k - \sigma_k)C_{hijs} = 0.$$

Permuting the indices j, s, k cyclically, adding the obtained equations to the preceding one and using (1.2) and (1.3), we get

$$(a_j - \sigma_j)C_{hisk} + (a_k - \sigma_k)C_{hijs} + (a_s - \sigma_s)C_{hijk} = 0. \tag{2.12}$$

Reversely, suppose that the vector field a_i and function σ satisfy the conditions (1.6) and (2.12). Then, the condition

$$(a_h - \sigma_h)C_{sijk} + (a_i - \sigma_i)C_{hsjk} + (a_s - \sigma_s)C_{ihjk} = 0$$

is satisfied, too, so that, adding, we can find

$$4(a_s - \sigma_s)C_{hijk} = 2(a_s - \sigma_s)C_{hijk} + \tag{2.13}$$

$$+ (a_h - \sigma_h)C_{sijk} + (a_i - \sigma_i)C_{hsjk} + (a_j - \sigma_j)C_{hisk} + (a_k - \sigma_k)C_{hijs}.$$

Transvecting (2.12) with g^{hs}, we get

$$(a_s - \sigma_s)C^s_{ijk} = 0,$$

which, because of (1.12), reduces to (2.7). Now, substituting (1.6) and (2.7) into (2.3), we have

$$\tilde{\nabla}_s\tilde{C}^h_{ijk} = 2(a_s - \sigma_s)C^h_{ijk} + (a^h - \sigma^h)C_{sijk} + (a_i - \sigma_i)C^h_{sjk} + \\ + (a_j - \sigma_j)C^h_{isk} + (a_k - \sigma_k)C^h_{ijs}.$$

This can, taking into account (2.13) and (2.2) be rewritten in the form

$$\tilde{\nabla}_s\tilde{C}^h_{ijk} = 4(a_s - \sigma_s)\tilde{C}^h_{ijk}.$$

Thus,

Theorem 4. *Let (M, g) be a CQR-manifold (1.6) and let (2.1) be a conformal change. Then $(M,\tilde{g})$ is a CR-manifold if and only if a condition (2.12) is satisfied and (2.11) is the corresponding recurrence vector.*

The conditions (2.11) and (2.12) show that $(M,\tilde{g})$ is, in fact, CRQR-manifold.

3. Examples of CQR-manifolds

Investigating CS-manifolds, Derdziński and Roter [4] proved that any essentially CS-manifold (i.e. a manifold which is neither conformally flat nor locally symetric), admits a unique function F (called the fundamental function) such that

$$FC_{hijk} = R_{ij}R_{hk} - R_{ik}R_{hj}.$$

Also, Derdziński proved [1] that if (M, g) is an essentially CS-manifold with a non-recurrent Ricci tensor and whose fundamental function F is a constant, then one of the following cases hold.

1. $F \neq 0$, rank$R_{ij} = 2$ andR_{ij} is semidefinite at each point of M;

2. $F \neq 0$, rank$R_{ij} = 2$ everywhere andR_{ij} is semidefinite at no point of M;

3. $F = 0$; then rank $R_{ij} \leq 1$ everywhere.

These three cases are called elliptic, hyperbolic and parabolic, respectively. Derdziński determined the metrics for all those manifolds ([1], [2], [3]). For example, he proved thet if (M, g) is an n-dimensional ($n \geq 4$) elliptic CS-manifold and $p \in M$ is a point at which $\nabla_k R_{ij} \neq 0$, there exists a coordinate system $(u^1, u^2, \ldots, u^n)$ in a neighbourthood of p such that $u^1(p) = \ldots = u^n(p) = 0$ and

$$(3.1) \qquad (g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & 0\ldots0 & 0 & g_{1n} \\ g_{21} & g_{22} & 0\ldots0 & g_{2n-1} & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & \vdots & g_{zx} & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & g_{n-12} & 0\ldots0 & 0 & 0 \\ g_{n1} & 0 & 0\ldots0 & 0 & 0 \end{pmatrix}$$

where x, y run over $3, \ldots, n-2$;

$$(3.2)\quad \begin{cases} g_{11} = 2u^n e^{-T} + 2u^{n-1} e^T \frac{\partial T}{\partial u^2} - \frac{1}{n-2}\varepsilon e^{2T} \sum_x \varepsilon_x (u^x)^2 \\ \qquad -(n-2)\varepsilon F^{-1} e^{2T}, \\ g_{22} = 2u^n (e^T \frac{\partial T}{\partial u^1} - e^{-T}) - \frac{1}{n-2}\varepsilon e^{2T} \sum_x \varepsilon_x (u^x)^2 \\ \qquad -(n-2)\varepsilon T^{-1} e^{2T}, \\ g_{12} = -u^n e^T \frac{\partial T}{\partial u^2} - u^{n-1}(e^T \frac{\partial T}{\partial u^1} + 2e^{-T}), \\ g_{1n} = g_{2n-1} = e^T, \ \ g_{xx} = \varepsilon_x, \ \ |\varepsilon_x| = 1. \end{cases}$$

T is a function of the first two variables u^1, u^2 and is given by

$$T = -\frac{1}{2}\log S,$$

where

$$2S(p) = \max\{\nabla_k R_{ij} v^i v^j v^k | v^i \in T_p(M), R_{ij} v^i v^j = \varepsilon, \varepsilon = \pm\},$$

and satisfies the quasi-linear elliptic partial equation

$$(3.3)\qquad \frac{\partial^2 T}{\partial u^1 \partial u^1} + \frac{\partial^2 T}{\partial u^2 \partial u^2} - 2e^{-4T} + \frac{1}{n-2}\varepsilon e^{2T} = 0.$$

Reversely, given the real number $F \neq 0, \varepsilon, \varepsilon_x$with$|\varepsilon| = |\varepsilon_x| = 1$ and a function T of two real variables (u^1, u^2) satisfyng (3.3). Then, (3.1) and (3.2) define an essentially CS-Riemannian metric with a fundamental function equal to F, which is elliptic (namely, its Ricci tensor is ε-semidefinite).

With respect to this coordinate system, the only non-zero components of the conformal curvature tensor are C_{1212} and C^n_{212}, C^{n-1}_{112}. Now let us consider the function

$$(3.4)\qquad \sigma = \sigma(u^1, \dots, u^{n-2}).$$

Then

$$\sigma_{n-1} = \frac{\partial \sigma}{\partial u^{n-1}} = 0, \sigma_n = \frac{\partial \sigma}{\partial u^n} = 0,$$

and

$$\sigma_x = \frac{\partial \sigma}{\partial u^x} \neq 0, x = 3, \dots, n-2.$$

Therefore,

$$\sigma_x C_{1212} + \sigma_1 C_{122x} + \sigma_2 C_{12x1} = \sigma_x C_{1212} \neq 0,$$

but

$$\sigma_r C^r_{212} = \sigma_n C^n_{212} = 0 \text{ and } \sigma_r C^r_{112} = \sigma_{n-1} C^{n-1}{}_{112} = 0.$$

So, condition (2.7) is satisfied while (2.8) is not. According to Theorem 3, the conformal change (2.1), (3.4) of the metric (3.1), (3.2), (3.3) is essentially CQR-metric. But if

$$\sigma = \sigma(u^1, u^2),$$

then condition (2.8) is satisfied too, and $(M, \bar{g})$ is CRQR-manifold.

Proceeding in a similar manner in the case of a CS-manifold of the hyperbolic type [2] or of the parabolic type [3], we can obtain new examples of essentially CQR-manifolds.

4. Totally umbilical hypersurface of conformally quasi-recurrent space

Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional Riemannian space covered by a system of coordinate neighbourhoods $\{U, y^\alpha\}$. Let (M, g) be a hypersurface of $\bar{M}$, defined in a locally coordinate system by means of a system of parametric equations $y^\alpha = y^\alpha(x^i)$. Here and in the sequel, Greek indices take values $1, 2, \ldots, n+1$ and Latin indices - the values $1, 2, \ldots, n$. Let N^α be a local unit normal to (M, g). Then, we have

$$g_{ij} = \bar{g}_{\alpha\beta} B^\alpha_i B^\beta_j, \tag{4.1}$$

$$\bar{g}_{\alpha\beta} N^\alpha B^\beta_i = 0, \bar{g}_{\alpha\beta} N^\alpha N^\beta = \varepsilon, \varepsilon = \pm 1 \tag{4.2}$$

and

$$B^\alpha_i B^\beta_j g^{ij} = \bar{g}^{\alpha\beta} - \varepsilon N^\alpha N^\beta, \tag{4.3}$$

where

$$B^\alpha_i = \frac{\partial y^\alpha}{\partial x^i}.$$

The (M, g) is called a totally umbilical hypersurface of the $(\bar{M}, \bar{g})$ if its second fundamental form h_{ij} satisfies.

$$h_{ij} = H g_{ij},$$

where H is a scalar function. For a totally umbilical hypersurface (M,g) of $(\bar{M},\bar{g})$, the equations of Gauss and Codazzi, respectively, can be written in the forms

$$\bar{R}_{\alpha\beta\gamma\delta}B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = R_{ijkl} - H^2(g_{il}g_{jk} - g_{ik}g_{jl}), \tag{4.4}$$

$$\bar{R}_{\alpha\beta\gamma\delta}N^\alpha B_j^\beta B_k^\gamma B_l^\delta = g_{jk}H_l - g_{jl}H_k. \tag{4.5}$$

Here, $\bar{R}_{\alpha\beta\gamma\delta}$ are the components of the curvature tensor of $(\bar{M},\bar{g})$ and $H_l = \nabla_l H$.

Contracting (4.4) with g^{il} and taking into account (4.3), we find

$$\bar{R_{\beta\gamma}}B_j^\beta B_k^\gamma = \varepsilon\bar{R}_{\alpha\beta\gamma\delta}N^\alpha B_j^\beta B_k^\gamma N^\delta + R_{jk} - \varepsilon(n-1)H^2 g_{jk}. \tag{4.6}$$

Contracting (4.6) with g^{jk} and using (4.3), we obtain

$$\bar{R_{\beta\gamma}}N^\beta N^\gamma = \tfrac{\varepsilon}{2}(\bar{R} - R) + \tfrac{n(n-1)}{2}H^2. \tag{4.7}$$

On the other hand, contracting (4.5) with g^{jk}, we find

$$\bar{R}_{\alpha\delta}N^\alpha B_l^\delta = (n-1)H_l. \tag{4.8}$$

Now, let us consider the conformal curvature tensor of the space $(\bar{M},\bar{g})$:

$$\bar{C}_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta} - \frac{1}{n-1}(\bar{R}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{R}_{\alpha\gamma}\bar{g}_{\beta\delta} +$$

$$+ \bar{R}_{\beta\gamma}\bar{g}_{\alpha\delta} - \bar{R}_{\beta\delta}\bar{g}_{\alpha\gamma}) + \frac{\bar{R}}{n(n-1)}(\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}).$$

Then we have [5]:

$$\bar{C}_{\alpha\beta\gamma\delta}N^\alpha B_j^\beta B_k^\gamma B_l^\delta = 0. \tag{4.9}$$

Also

$$\bar{C}_{\alpha\beta\gamma\delta}N^\alpha B_j^\beta B_k^\gamma N^\delta = \bar{R}_{\alpha\beta\gamma\delta}N^\alpha B_j^\beta B_k^\gamma N^\delta -$$

$$- \frac{1}{n-1}(\bar{R}_{\alpha\delta}N^\alpha N^\delta g_{jk} + \varepsilon\bar{R}_{\beta\gamma}B_j^\beta B_k^\gamma) + \frac{\bar{R}}{n(n-1)}\varepsilon g_{jk}.$$

Substituting (4.6) and (4.7), we obtain

$$\bar{C}_{\alpha\beta\gamma\delta}N^{\alpha}B_j^{\beta}B_k^{\gamma}N^{\delta} = \varepsilon\frac{n-2}{n-1}\bar{R}_{\beta\gamma}B_j^{\beta}B_k^{\gamma} - \varepsilon R_{jk} +$$

$$+ [-\frac{n-2}{2n(n-1)}\varepsilon\bar{R} + \frac{1}{2(n-1)}\varepsilon R + \frac{n-2}{2}H^2]g_{jk}.$$

This can be rewritten in the form

(4.10) $$\tfrac{1}{n-1}\bar{R}_{\beta\gamma}B_j^{\beta}B_k^{\gamma} = \tfrac{1}{n-2}Q_{jk} + \tfrac{1}{n-2}R_{jk} +$$

$$+ [\frac{1}{2n(n-1)}\bar{R} - \frac{1}{2(n-1)(n-2)}R - \frac{\varepsilon}{2}H^2]g_{jk},$$

where we have put

(4.11) $$\bar{C}_{\alpha\beta\gamma\delta}N^{\alpha}B_j^{\beta}B_k^{\gamma}N^{\delta} = \varepsilon Q_{jk}.$$

Substituting (4.10) into

$$\bar{C}_{\alpha\beta\gamma\delta}B_i^{\alpha}B_j^{\beta}B_k^{\gamma}B_l^{\delta} = \bar{R}_{\alpha\beta\gamma\delta}B_i^{\alpha}B_j^{\beta}B_k^{\gamma}B_l^{\delta} -$$

$$- \frac{1}{n-1}(\bar{R}_{\alpha\delta}B_i^{\alpha}B_l^{\delta}g_{jk} - \bar{R}_{\alpha\gamma}B_i^{\alpha}B_k^{\gamma}g_{jl} + \bar{R}_{\beta\gamma}B_j^{\beta}B_k^{\gamma}g_{il} - \bar{R}_{\beta\delta}B_j^{\beta}B_l^{\delta}g_{ik}) +$$

$$+ \frac{\bar{R}}{n(n-1)}(g_{il}g_{jk} - g_{ik}g_{jl}),$$

and taking into account (4.4), we find

(4.12) $$\bar{C}_{\alpha\beta\gamma\delta}B_i^{\alpha}B_j^{\beta}B_k^{\gamma}B_l^{\delta} = C_{ijkl} - \tfrac{1}{n-2}(Q_{il}g_{jk} - Q_{ik}g_{jl} + Q_{jk}g_{il} - Q_{jl}g_{ik}),$$

where $C_{ijkl}$is the conformal curvature tensor of the hypersurface (M, g).

We note that

(4.13) $$Q_{jk} = Q_{kj} \text{ and } Q_{jk}g^{jk} = 0,$$

and

(4.14) $$\bar{C}_{\alpha\beta\gamma\delta}B_i^{\alpha}B_j^{\beta}B_k^{\gamma}B_l^{\delta}g^{il} = -Q_{jk}.$$

Let ∇ be the operator of the van der Wearden-Bortolotti covariant derivative. Then

$$\nabla_r B_i^\beta = \varepsilon h_{rj} N^\beta.$$

Applying the operator ∇ to (4.12) and using (4.9), we have

$$\nabla_\rho \bar{C}_{\alpha\beta\gamma\delta} B_r^\rho B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = \nabla_r C_{ijkl} -$$

$$(4.15) \qquad - \tfrac{1}{n-2}(g_{jk}\nabla_r Q_{il} - g_{jl}\nabla_r Q_{ik} + g_{il}\nabla_r Q_{jk} - g_{ik}\nabla_r Q_{jl}).$$

Now, let us suppose that $(\bar{M}, \bar{g})$ is conformally quasi-recurrent. Then,

$$\bar{\nabla}_\rho \bar{C}_{\alpha\beta\gamma\delta} = 2a_\rho \bar{C}_{\alpha\beta\gamma\delta} + a_\alpha \bar{C}_{\rho\beta\gamma\delta} + a_\beta \bar{C}_{\alpha\rho\gamma\delta} + a_\gamma \bar{C}_{\alpha\beta\rho\delta} + a_\delta \bar{C}_{\alpha\beta\gamma\rho},$$

and we have

$$\bar{\nabla}_\rho \bar{C}_{\alpha\beta\gamma\delta} B_r^\rho B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = 2a_\rho B_r^\rho \bar{C}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta +$$

$$+ a_\alpha B_i^\alpha \bar{C}_{\rho\beta\gamma\delta} B_r^\rho B_j^\beta B_k^\gamma B_l^\delta + a_\beta B_j^\beta \bar{C}_{\alpha\rho\gamma\delta} B_i^\alpha B_r^\rho B_k^\gamma B_l^\delta +$$

$$+ a_\gamma B_k^\gamma \bar{C}_{\alpha\beta\rho\delta} B_i^\alpha B_j^\beta B_r^\rho B_l^\delta + a_\delta B_l^\delta \bar{C}_{\alpha\beta\gamma\rho} B_i^\alpha B_j^\beta B_k^\gamma B_r^\rho.$$

As for vector field a_α, it can be decomposed, at the points of (M, g), as follows

$$(4.16) \qquad a_\alpha = \bar{g}_{\alpha\mu} a^\mu = \bar{g}_{\alpha\tau}(B_t^\tau a^t + aN^\tau).$$

Substituting this, (4.15) and (4.12) into the preceding relation and putting $a_r = g_{rt}a^t$, we find

$$(4.17) \qquad \nabla_r C_{ijkl} = 2a_r C_{ijkl} + a_i C_{rjkl} + a_j C_{irkl} + a_k C_{ijrl} + a_l C_{ijkr} +$$

$$+ \frac{1}{n-2}\{(g_{jk}\nabla_r Q_{il} - g_{jl}\nabla_r Q_{ik} + g_{il}\nabla_r Q_{jk} - g_{ik}\nabla_r Q_{jl}) -$$

$$- 2a_r(g_{jk}Q_{il} - g_{jl}Q_{ik} + g_{il}Q_{jk} - g_{ik}Q_{jl}) -$$

$$- a_i(g_{jk}Q_{rl} - g_{jl}Q_{rk} + g_{rl}Q_{jk} - g_{rk}Q_{jl}) -$$

$$- a_j(g_{rk}Q_{il} - g_{rl}Q_{ik} + g_{il}Q_{rk} - g_{ik}Q_{rl}) -$$
$$- a_k(g_{jr}Q_{il} - g_{jl}Q_{ir} + g_{il}Q_{jr} - g_{ir}Q_{jl}) -$$
$$- a_l(g_{jk}Q_{ir} - g_{jr}Q_{ik} + g_{ir}Q_{jk} - g_{ik}Q_{jr})\}.$$

The space $(\bar{M}, \bar{g})$ being conformally quasi-recurrent, the condition of the form (1.12) is satisfied, i.e. $a^\rho \bar{C}_{\rho\alpha\beta\gamma} = 0$. Thus,

$$a^\rho \bar{C}_{\rho\beta\gamma\delta} B_j^\beta B_k^\gamma B_l^\delta = 0$$

which can, taking into account (4.16) and (4.9), be rewritten in the form

$$a^r \bar{C}_{\rho\beta\gamma\delta} B_r^\rho B_j^\beta B_k^\gamma B_l^\delta = 0.$$

Substituting (4.12) into this equation, we find

$$a_r C^r_{jkl} = \tfrac{1}{n-2} a^r (g_{jk}Q_{rl} - g_{jl}Q_{rk} + g_{rl}Q_{jk} - g_{rk}Q_{jl}). \tag{4.18}$$

Contracting (4.18) with g^{jk} and using (4.13). we get

$$a^r Q_{rj} = 0. \tag{4.19}$$

Therefore, (4.18) reduces to

$$a^r C_{rjkl} = \tfrac{1}{n-2}(a_l Q_{jk} - a_k Q_{jl}). \tag{4.20}$$

On the other hand, contracting (4.17) with g^{jk}, and using (1.2). (4.13), (4.19) and (4.20), we have

$$\nabla_r Q_{il} = 2a_r Q_{il} + a_i Q_{rl} + a_l Q_{ir}.$$

Substituting this into (4.17), we find

$$\nabla_r C_{ijkl} = 2a_r C_{ijkl} + a_i C_{rjkl} + a_j C_{irkl} + a_k C_{ijrl} + a_l C_{ijkr} +$$

$$+ \tfrac{1}{n-2}[g_{rl}(a_j Q_{ik} - a_i Q_{jk}) + g_{rk}(a_i Q_{jl} - a_j Q_{il}) + \tag{4.21}$$

$$+ g_{rj}(a_l Q_{ik} - a_k Q_{il}) + g_{ri}(a_k Q_{jl} - a_l Q_{jk})].$$

Thus, if

$$a_j Q_{ik} = a_i Q_{jk} \tag{4.22}$$

then (M,g) is conformally quasi-recurrent. Conversely, if (M,g) is conformally quasi-recurrent, then (4.21) reduces to

$$g_{rl}(a_j Q_{ik} - a_i Q_{jk}) + g_{rk}(a_i Q_{jl} - a_j Q_{il}) +$$

$$g_{rj}(a_l Q_{ik} - a_k Q_{il}) + g_{ri}(a_k Q_{jl} - a_l Q_{jk}) = 0,$$

from which, contracting with g^{rl}, we obtain (4.22). Thus, we have

Theorem 5. *Let $(\bar{M},\bar{g})$ be CQR-manifold and let (M,g) be its totally umbilical hypersurface. The (M,g) is CQR-manifold if and only if the vector field $a_i = a_\alpha B_i^\alpha$ satisfies (4.22). If $a_i = 0$, i.e. if at the points of (M,g), the vector field a^α is orthogonal to (M,g), (M,g) is CS-manifold.*

Now, let us suppose that $(\bar{M}\bar{g})$ is CRQR-manifold. Then, the condition

$$a_\delta \bar{C}_{\alpha\beta\gamma\delta} + a_\gamma \bar{C}_{\alpha\beta\delta\sigma} + a_\delta \bar{C}_{\alpha\beta\sigma\gamma} = 0 \tag{4.23}$$

is satisfied, too, Contracting it with $N^\sigma B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta$ and using (4.9), we find

$$a_\sigma N^\sigma \bar{C}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = 0.$$

If, at the points of (M,g), vector filed a^α satisfies $a_\sigma N^\sigma \neq 0$ the preceding relation reduces to

$$\bar{C}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = 0,$$

from which, using (4.14), we obtain $Q_{jk} = 0$. Therefore, (4.12) reduces to $C_{ijkl} = 0$.

On the other hand, contracting (4.23) with $B_s^\sigma B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta$ we get

$$a_s \bar{C}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta + a_k \bar{C}_{\alpha\beta\delta\sigma} B_i^\alpha B_j^\beta B_l^\delta B_s^\sigma +$$

$$+ a_l \bar{C}_{\alpha\beta\sigma\gamma} B_i^\alpha B_j^\beta B_s^\sigma B_k^\gamma = 0$$

which, in view, of (4.12), can be written in the form

$$a_s C_{ijkl} + a_k C_{ijls} + a_l C_{ijsk} + \tag{4.24}$$

$$+ \frac{1}{n-2}[g_{jl}(a_s Q_{ik} - a_k Q_{is}) + g_{js}(a_k Q_{il} - a_l Q_{ik}) + g_{jk}(a_l Q_{is} - a_s Q_{il}) +$$
$$+ g_{ik}(a_s Q_{jl} - a_l Q_{js}) + g_{il}(a_k Q_{js} - a_s Q_{jk} + g_{is}(a_l Q_{jk} - a_k Q_{jl})] = 0$$

According to Theorem 5, the hypersurface is conformally quasi-recurrent if and only if condition (4.22) is satisfied. But in this case, (4.24) reduces to

$$a_s C_{ijkl} + a_k C_{ijls} + a_l C_{ijsk} = 0.$$

This means that the hypersurface is CRQR-manifold. Thus, we have

Theorem 6. *Let $(\bar{M}, \bar{g})$ be a CRQR-manifold and let (M, g) be its totally unbilical hypersurface. If at the points of (M, g) the vector field a^α is not tangetial to (M, g), (M, g) is conformally flat. If at the points of (M, g) the vector field a^α is tangential to (M, g) and (M, g) is a CQR-manifold, it is a CRQR-manifold, too.*

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REZIME

PRIMEDBA O KONFORMNO KVAZI REKURENTNIM MNOGOSTRUKOSTIMA

Ispitani su uslovi pod kojima se konformno kvazi rekurentna mnogostrukost svodi na konformno rekurentnu. Dati su primeri konformno kvazi rekurentnih prostora koji nisu ni konformno simetrični ni konformno rekurentni. Ispitane su totalno ombilične hiperpovrši konformno kvazi rekurentnih prostora.

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