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ON THE NUMERICAL SOLUTION OF A SINGULARLY PERTURBED NONLOCAL PROBLEM

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Abstract

The numerical solution of a linear singularly pertubed nonlocal problem is considered. To approximate the differential equation, the Hermitian scheme on a special nonuniform mesh is used. The fourth order convergence uniform in the perturbation parameter is proved. The numerical results show the pointwise convergence, too.

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1. Introduction

In this paper we shall consider the problem

$$
\begin{aligned}
&\varepsilon^2 u'' + b(x)u = f(x), \quad x \in I = [0,1], \\
(1) \qquad &u(0) = 0 \\
&u(1) = \sum_{i=1}^{m} c_i u(s_i) + d, \\
&d, c_i \in \mathbf{R},\ i = 1,2,\ldots m, \quad s_i \in (0,1),\ i = 1,2,\ldots,m
\end{aligned}
$$

where, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 << 1$ is a small perturbation parameter. We assume that the following conditions are satisfied:

$$b \in C^k(I),\ k \in \mathbf{N},\ 0 < \beta^2 \le b(x),\ x \in I \tag{2}$$

$$-\infty < \sum_{i=1}^{m} c_i \omega_0(s_i) < 1, \tag{3}$$

where

$$\omega_0(x) = \frac{(\exp(\alpha_0 x) - \exp(-\alpha_0 x))}{\exp(\alpha_0) - \exp(-\alpha_0))}, \quad \alpha_0 = \frac{\beta}{\varepsilon}.$$

Some problems of type (1)-(2) arise as a model problem for some physical phenomena, see [2], [9]. In [2] the numerical solution of problem (1) under conditions (2)-(3) was given. The finite elements on an equidistant mesh was applied there and second order uniform convergence was obtained. The case $\sum_{i=1}^{m} |c_i| < 1$ was considered in [8], too.

In this paper we use a uniform fourth order difference scheme developed in [5]. This sheme is constructed on a special nonequidistant meshes. The reason for using these meshes is our aim to obtain more mesh points in the region of boundary layers, whose width is $O(\varepsilon)$. It is known, see [2], that conditions (2)-(3) imply that problem (1) has a uniqe solution u_ε for which one can estimate the derivates, see [3]. In general, the solution u_ε has boundary

layers at $x = 0$ and $x = 1$. So, knowing the behaviour of the exact solution of our problem, we use the mesh generating function $\lambda(t)$ which is suitable for problem (1).

We end our paper with some numerical results, which show that theoretical order of convergence is also established numerically.

The constans M will be independent of the discretization mesh and ε.

2. The numerical method

We shall consider discretization on discretization mesh $I_s = I_h\ U\{s_1, s_2, \ldots \ldots, s_m\}$ where

$$I_h = \{x_i = \lambda(t_i) : i = 0, 1, \ldots, n\},\ t_i = ih,\ h = \frac{1}{n},\ n = 2n_0, n_0 \in \mathbf{N},$$

with mesh generating function [10], [11], [5]:

$$\lambda(t) = \begin{cases} \omega(t) = \frac{a\varepsilon t}{q-t}, & t \in [0,\alpha], \\ \omega(\lambda) + \omega'(\alpha)(t-\alpha), & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1], \end{cases}$$

and

$$q \in (0, 0.5), \ \ a\varepsilon_0 \le q.$$

The parameter α is

$$\alpha = \frac{q - \sqrt{aq\varepsilon(1 - 2q + 2a\varepsilon)}}{1 + 2a\varepsilon}.$$

It is easy to see that $\lambda(t)$ is a monotone increasing function on I, and we can consider the points s_i as values of $\lambda(t_{si})$, for some $t_{si} \in (0,1)$, $i = 1, 2, \dots, m$. From now on we denote the points of the mesh I_s as x_i, $i = 0, 1, \dots N$, $n \le N \le n + m$.

The other properties of mesh generating function $\lambda(t)$, which are important for our analysis, are given in [5], [10], [11], [12].

Let

$$Q = 2(1 + \frac{\sqrt{3}}{3})$$

$$I_h' = \{x_i \in I_h : q - Qh < t_{i-1} < \alpha \ \text{ or } \ 1 - \alpha < 1 - Qh\}.$$

Let us note that the set I_h' can be empty.

In order to obtain the numerical solution of problem (1) we shall consider as in [3] the following two boundary value problems

$$-\varepsilon^2 z'' + b(x) z = 0, \qquad (4)$$

$$z(0) = 0, \ z(1) = 1,$$

$$-\varepsilon^2 y'' + b(x) y = f(x), \qquad (5)$$

$$y(0) = 0, y(1) = 0.$$

In [3] it is proved that

$$u_\varepsilon(x) = y_\varepsilon(x) + \lambda_\varepsilon z(x), \quad \lambda_\varepsilon = \frac{\sum_{i=1}^m c_i y_\varepsilon(s_i) + d}{1 - \sum_{i=1}^m c_i z_\varepsilon(s_i)}$$

is the solution of problem (1), where z_ε and y_ε are the solution to problems (4) and (5) respectively.

Since, see [3], $0 \le z(x) \le \omega_0(x)$, λ_ε is well defined and bounded.

If we have numerical solutions of the problems (4) and (5), say z_h and y_h, we can form in the same way the numerical solution u_h of (1):

$$(6) \qquad u_h = y_h + \lambda_h z_h, \quad \lambda_h = \frac{\sum_{i=1}^m c_i y_h(s_i) + d}{1 - \sum_{i=1}^m c_i z_h(s_i)}.$$

Let us consider now problem (4). Problem (5) can be treated in the same way. We discretize problem (4) on the mesh I_s by using the same scheme as in [5], [6], [7]:

$$(7) \qquad \begin{gathered} z_h(0) = 0 \\ \varepsilon^2(a_1(i)z_h(x_{i-1}) + a_0(i)z_h(x_i) + a_2(i)z_h(x_{i+1})) + \\ +B_1(i)b(x_{i-1}) + B_0 b(x_i) + B_2 b(x_{i+1}) = 0, \quad i = 1,2,\dots,N-1, \\ z_h = 1, \end{gathered}$$

where

$$a_1(i) = \frac{-2}{h_i(h_i + h_{i+1})}, \quad a_0(i) = \frac{2}{h_i h_{i+1}}, \quad a_2(i) = \frac{-2}{h_{i+1}(h_i + h_{i+1})},$$

$$h_i = x_i - x_{i-1}, \quad i = 1,2,\dots,n,$$

and, if $x_i \in I_s \,/\, I'_h$,

$$B_1(i) = \frac{-a_1(i)}{12}(h_i^2 - h_{i+1}^2 + h_i h_{i+1}),$$

$$B_0(i) = \frac{-a_0(i)}{12}(h_i^2 + h_{i+1}^2 + 3h_i h_{i+1}),$$

$$B_2(i) = \frac{-a_2(i)}{12}(h_{i+1}^2 - h_i^2 + h_i h_{i+1}),$$

or

$$B_0(i) = 1, \quad B_1(i) = B_2(i), \quad \text{if } x_i \in I'_h.$$

Now, in the same way as in [5] we can prove the following theorem.

Theorem 1. *Suppose that condition (2) is satisfied with* $k = 8$. *Let* $|b'(x)| \leq L$, $x \in I$ *and let* z_ε *be the solution of (4) and* $z_{\varepsilon,h} = [z_\varepsilon(0),\ z_\varepsilon(x_1),\ \ldots, z_\varepsilon(1)]^T$. *If*

$$n > \max\{\frac{2L}{3\beta^2(1-2q)K},\ \frac{Q}{q}\},\ \ 0 < K < 1, \tag{8}$$

then there exists a unique solution $z_h = [z_h(0),\ z_h(x_1), \ldots, z_h(1)]^T$ *to (7) and it holds*

$$\|z_{\varepsilon,h} - z_h\|_\infty \leq Mh^4. \tag{9}$$

Analogously, under the same assumptions by using same scheme, we obtain that there exists a uniqe numerical solution $y_h = [y_h(0), y_h(x_1), \ldots$ $\ldots, y_h(1)]^T$ *and that it holds that*

$$\|y_{\varepsilon,h} - y_h\|_\infty \leq Mh^4, \tag{10}$$

where y_ε *is the solution to problem (5) and* $y_{\varepsilon,h} = [y_\varepsilon(0), y_\varepsilon(x_1), \ldots, y_\varepsilon(1)]^T$.

Now, we can prove the main result.

Theorem 2. *Suppose that conditions (2), (3) with* $k = 8$ *are satisfied. Let* $|b'(x)| \leq L$, $x \in I$ *and let* z_h *and* y_h *be the numerical solutions to (4) and (5) respectively on discretization mesh* I_s *with (8). Then, if*

$$1 - \sum_{i=1}^{m} c_i z_h(s_i) \neq 0 \tag{11}$$

it holds that

$$|\ u_{\varepsilon,h} - u_h\|_\infty \leq Mh^4, \tag{12}$$

where u_h *is given in (6),* $u_{\varepsilon,h} = [u_\varepsilon(0),\ u_\varepsilon(x_1), \ldots, u_\varepsilon(1)]^T$ *and* u_ε *is the solution to (1).*

Proof. In [3] it is proved that under considered conditions there exists a unique solution u_ε to problem (1). Also, it is shown that $|\lambda|$ is bounded.

From [1], [4] it follows that z_ε and y_ε are bounded. Also, from (9) and (10) we conculde that z_h and y_h are bounded. Obviously

$$\|u_{\varepsilon,h} - u_h\|_\infty \le \|y_{\varepsilon,h} - y_h\|_\infty + \tag{13}$$
$$|\lambda_\varepsilon|\,\|z_{\varepsilon,h} - z_h\|_\infty + |\lambda_\varepsilon - \lambda_h|\,\|z_h\|_\infty .$$

From the definitiones of λ_ε and λ_h we obtain

$$\lambda_\varepsilon - \lambda_h = \sum_{i=1}^{m} c_i(y_\varepsilon(s_i) - y_h(s_i)) + \sum_{i=1}^{m} c_i(\lambda_\varepsilon z_h(s_i) - \lambda_h z_h(s_i))$$

and thus

$$(\lambda_\varepsilon - \lambda_h)(1 - \sum_{i=1}^{m} c_i z_\varepsilon(s_i)) = \sum_{i=1}^{m} c_i[y_\varepsilon(s_i) - y_h(s_i) + \lambda_h(z_\varepsilon(s_i) - z_h(s_i))],$$

$$|\lambda_\varepsilon - \lambda_h| \le M(\|y_{\varepsilon,h} - y_h\|_\infty + \|z_{\varepsilon,h} - z_h\|_\infty) \le Mh^4.$$

Now, from (13) follows (12), that is, we have a numerical solution to problem (1) which is of the fourth order accuarcy uniform in ε. □

Remark. Condition (11) is an artificial one. In effect, z_h is the fourth order approximation of z and for a sufficiencly small h from (3) follows (11).

3. Numerical examples

As illustrative test problems to verify our method we consider well-known problems, [5-7], [10-12]:

$$-\varepsilon u'' + u = 1 \tag{14}$$

$$-\varepsilon^2 u'' + u + \cos^2(\pi x) = -2(\varepsilon\pi)^2\cos(2\pi x) \tag{15}$$

with the conditions

$$u(0) = 0, u(1) = \sum_{i=1}^{m} c_i u(s_i) + d. \tag{16}$$

The exact solutions of these problems in case $u(0) = u(1) = 0$, are given in [5-6]. Now, for given m, c_i, $i = 1, 2, \ldots, m$, we determine d so that $u(1) = 0$.

We denote by E_N the maximum of $\mid u_\varepsilon(x) - u_h(x) \mid$, $x \in I_s$, i.e.

$$E_N = \|u_{\varepsilon,h} - u_h\|_\infty .$$

Also, we define in the usual way the order of convergence Ord for two successive values of n with respective errors E_N and E_{N_2}:

$$\mathrm{Ord} = \frac{\log E_N - \log E_{N_2}}{\log N - \log N_2}.$$

We expect that Ord = 4. Table 1 presents the results for the numerical solution obtained by our method for example (14), (16), where $a = 1$, $q = 0.48$, $m = 3$ and

i	s_i	c_i
1	0.99999	0.0300
2	0.10000	0.2000
3	0.20000	5.0000

Table 1.

$N(n)\backslash\varepsilon$	2^{-10}	2^{-15}	2^{-20}	$2^{-30}-2^{-50}$	
7(2)	3.1959E-01	3.3668E-01	3.3722E-01	3.3724E-01	E_N
	-	-	-	-	Ord
11(4)	5.9696E-02	5.9980E-02	5.9989E-02	5.9989E-02	E_N
	3.7120	3.8168	3.8200	3.8201	Ord
19(8)	9.4931E-03	9.4948E-03	9.4949E-03	9.4949E-03	E_N
	3.3642	3.3726	3.3729	3.3729	Ord
35(16)	2.9128E-03	2.9128	2.9128E-03	2.9128E-03	E_N
	1.9339	1.9342	1.9342	1.9342	Ord
67(32)	1.5315E-04	1.5315E-04	1.5315E-04	1.5315E-04	E_N
	4.5360	4.5360	4.5360	4.5360	Ord
131(64)	9.1652E-06	9.1652E-06	9.1652E-06	9.1652E-06	E_N
	4.1999	4.1999	4.1999	4.1999	Ord
259(128)	5.6859E-07	5.6859E-07	5.6859E-07	5.6859E-07	E_N
	4.0785	4.0785	4.0785	4.0785	Ord
515(256)	3.5390E-08	3.5390E-08	3.5390E-08	3.5390E-08	E_N
	4.0398	4.0398	4.0398	4.0398	Ord

In Table 2 are given the results for example (15), (16) where $\varepsilon = 2^{-25}$, $a = 1$, $q = 0.4$ and s_i, c_i, $i = 1,2,3$, are as above. Here T is the number of mesh points in $(0,\varepsilon) \cup (1-\varepsilon, 1)$.

Table 2.

n	N	T	E_N	Ord	λ_h
2	7	1	1.8888E-01	8.5650E-01	3.546631E-09
4	11	3	1.2686E-01	8.8056E-01	-1.715722E+00
8	19	7	1.4140E-02	4.0145E+00	4.865624E-01
16	35	13	2.9822E-03	2.5476E+00	2.129330E+00
32	67	25	3.3565E-04	3.3639E+00	2.844466E+00
64	131	51	1.9299E-05	4.2595E+00	3.173170E+00
128	259	103	1.1801E-06	4.0996E+00	3.330554E+00
265	515	205	7.3429E-08	4.0403E+00	3.407553E+00
512	1027	409	4.5861E-09	4.0179E+00	3.445636E+00

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REZIME

O NUMERIČKOM REŠAVANJU SINGULARNO PERTURBOVANOG NELOKALNOG PROBLEMA

Posmatra se numeričko rešavanje linearnog singularno perturbovanog nelokalnog problema. Koristi se specijalna mreža diskretizacije i diferencna šema zasnovana na Hermitovoj aproksimaciji diferencijalne jednačine. Dokazana je uniforma po malom parametru konvergencija četvrtog reda. Numerički rezultati takođe pokazuju tačkastu konvergenciju.

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