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SPECIAL ELEMENTS OF THE LATTICE AND LATTICE IDENTITIES

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Abstract

Special elements of a lattice L (distributive, codistributive, neutral etc.) induce a congruence relation on L. Here we consider the following problem: If the lattice identity is satisfied on a class of that congruence (a sublattice of L), under which conditions this identity holds on the lattice itself?

Several algebraic results are deduced from the obtained lattice properties: characterizations of the congruence extension and of the congruence intersection properties, and some general properties of the weak congruence lattice of an algebra.

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1. Preliminaries

In the paper [3], it was proved that the weak congruence lattice Cw$\mathcal{A}$ of an algebra $\mathcal{A}$ (i.e. the lattice of all the congruences on all the subalgebras of $\mathcal{A}$ is modular if and only if both Con$\mathcal{A}$ and Sub$\mathcal{A}$ are modular lattices, and $\mathcal{A}$ has the CEP and the CIP (the congruence extension, and the congruence intersection property). Con$\mathcal{A}$ and Sub$\mathcal{A}$ are two sublattices of Cw$\mathcal{A}$. Thus, under some conditions (the CEP and the CIP), a lattice identity (modularity) holds on Cw$\mathcal{A}$ if and only if it holds on two special sublattices of that lattice. To consider the general problem, concerning any lattice identity on any algebraic lattice L, provided that this identity holds on the special sublattices of L, we need the following definitions and propositions, which are mainly from [1], [2], and from the references given there.

An element a of the lattice L is **codistributive**, if for all $x, y \in L$

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y),$$

or, equivalently, if and only if the mapping $m_a : x \to x \wedge a$ is a homomorphism of L into the ideal (a].

Note that for a codistributive element a of L, the filter [a) is the class of the congruence induced by m_a.

A **distributive** element a of L, is defined dually (the corresponding homomorphism from L into the filter [a) is obviosly $x \to x \vee a$, and the ideal (a] is the class of the congruence induced by that mapping).

An element a of L is said to be **comodular**, if $x \leq y$ implies $x \vee (a \wedge y) = (x \vee a) \wedge y$.

The lattice Cw$\mathcal{A}$ of weak congruences of an algebra $\mathcal{A} = (A, F)$ is the lattice of all the symmetric and transitive subalgebras of $\mathcal{A}^2$ under the set inclusion. The diagonal relation $\Delta = \{(x, x) \mid x \in A\}$ is a codistributive element of that lattice , $[\Delta)$ is Con$\mathcal{A}$, and Sub$\mathcal{A}$ is isomorphic with $(\Delta]$ under $\rho \to \{x \mid x\rho x\}$.

Recall that $\mathcal{A}$ has the **congruence extension property** (CEP), if every congruence on a subalgebra of $\mathcal{A}$ is a restriction of the congruence on $\mathcal{A}$.

$\mathcal{A}$ has the CEP if and only if Δ is a comodular element of Cw$\mathcal{A}$ ([2]).

$\mathcal{A}$ is said to have the **congruence intersection property** (CIP), if for $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$, $\mathcal{B},\mathcal{C} \in Sub\mathcal{A}$,

$$(\rho \cap \theta)_A = \rho_A \cap \theta_A,$$

where ρ_A stands for the least congruence on $\mathcal{A}$ extending ρ. Since $\rho_A = \rho \vee \Delta$ in Cw$\mathcal{A}$, $\mathcal{A}$ has the CIP if and only if Δ is a distributive element of Cw$\mathcal{A}$.

2. Identities on lattices

In this section we shall describe the conditions under which a lattice identity holds on L, provided that it holds on (a] and (or) on [a), where a is a special element of L (distributive, codistributive, modular, etc.).

Lemma 1. *If a is a distributive element of the lattice L, and $f(x_1, \ldots, x_n)$ is an arbitrary lattice therm, then for all $x_1, \ldots, x_n \in L$,*

$$f(x_1, \ldots, x_n) \vee a = f(x_1 \vee a, \ldots, x_n \vee a).$$

Proof. Straightforward, by induction of the number of operational symbols.

□

A dual proposition is the following.

Lemma 2. *If a is a codistributive element of the lattice L, and $f(x_1, \ldots, x_n)$ is an arbitrary lattice therm, then for all $x_1, \ldots, x_n \in L$,*

$$f(x_1, \ldots, x_n) \wedge a = f(x_1 \wedge a, \ldots, x_n \wedge a).$$

The following three lemmas will be necessary in the algebraic applications of the results in this section. We shall assume that the classes of the congruence induced by the mapping $x \to x \wedge a$, where a is a codistributive element of the lattice L, always have the greatest elements. For the class to which $x \in L$ belongs, the greatest element will be denoted by $\bar{x}$.

Lemma 3. *If a is a codistributive element of the lattice L, then the following are equivalent, for all $x, y \in L$:*

(i) if $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$, then $x = y$;

(ii) if $x \leq y$, then $x \vee (a \wedge y) = (x \vee a) \wedge y$;

(iii) if $x \leq \bar{y}$, then $x \vee (a \wedge \bar{y}) = (x \vee a) \wedge \bar{y}$;

(iv) $x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y)$.

Proof. $(i) \iff (ii) \iff (iii)$, proved in[2].
$(iv) \Rightarrow (ii)$, obvious.
$(ii) \Rightarrow (iv)$. Since $x \leq x \vee y$, it follows that

$$x \vee (a \wedge (x \vee y)) = (x \vee a) \wedge (x \vee y).$$

Moreover, a is codistributive, and

$$x \vee (a \wedge (x \vee y)) = x \vee (a \wedge x) \vee (a \wedge y) = x \vee (a \wedge y).$$

□

Lemma 4. *If a is a distributive element of the lattice L, then the following are equivalent, for all $x, y \in L$:*

(i) if $x \leq y$, then $x \vee (a \wedge y) = (x \vee a) \wedge y$;

(ii) $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$.

Proof. $(ii) \Rightarrow (i)$, obvious.
$(i) \Rightarrow (ii)$. Since $x \wedge y \leq x$, it follows that

$$(x \wedge y) \vee (a \wedge y) = ((x \wedge y) \vee a) \wedge y, \text{ and}$$

$$((x \wedge y) \vee a) \wedge y = (x \vee a) \wedge (y \vee a) \wedge y = (x \vee a) \wedge y.$$

□

Lemma 5. *If a is a codistributive element of the lattice L, satisfying any of the conditions from Lemma 3. then the following are equivalent :*

(i) a is a distributive element of L ;

(ii) for all $x, y \in L, (x \wedge a) \vee (x \wedge y) = ((x \wedge a) \vee y) \wedge x$.

Proof. $(i) \Rightarrow (ii)$

$$a \wedge ((x \wedge a) \vee (x \wedge y)) = (a \wedge x \wedge a) \vee (a \wedge x \wedge y) = a \wedge x.$$

$$a \wedge (((x \wedge a) \vee y) \wedge x) = a \wedge x.$$

$$a \vee ((x \wedge a) \vee (x \wedge y)) = a \vee (x \wedge y).$$

$$a \vee (((x \wedge a) \vee y) \wedge x) = (a \vee (x \wedge a) \vee y) \wedge (a \vee x) = (a \vee y) \wedge (a \vee x) = a \vee (x \wedge y).$$

Now, by (i) in Lemma 3, it follows that

$$(x \wedge a) \vee (x \wedge y) = ((x \wedge a) \vee y) \wedge x.$$

$(ii) \Rightarrow (i)$. We shall use condition (iii) from Lemma 3. Therefore, let $\bar{z} = \overline{x \vee y}$. Hence, $\bar{z} \wedge a = (x \vee y) \wedge a$, and thus

$$\bar{z} \wedge a = (x \wedge a) \vee (y \wedge a), \text{ and } \bar{z} \wedge a \geq x \wedge a, \ \bar{z} \wedge a \geq y \wedge a, \ \bar{z} \geq x, \ \bar{z} \geq y.$$

Now, we have

$$a \vee (x \wedge y) = a \vee (y \wedge a) \vee (x \wedge y) = a \vee (((x \wedge a) \vee y) \wedge x) =$$
$$= a \vee (((x \wedge a) \vee (y \wedge a) \vee y) \wedge x) = a \vee ((((x \vee y) \wedge a) \vee y) \wedge x) =$$
$$= a \vee (((\bar{z} \wedge a) \vee y) \wedge x) = a \vee (\bar{z} \wedge (a \vee y) \wedge x) = a \vee (x \wedge (a \vee y)) =$$
$$((a \vee y) \wedge a) \vee (x \wedge (a \vee y)) = (a \vee x) \wedge (a \vee y).$$

□

Theorem 1. *If a is a distributive, codistributive and comodular element of the lattice L, then an arbitrary lattice identity is satisfied on L, if and only if this identity holds on $(a]$ and on $[a)$.*

Proof.
($\Rightarrow$). Obvious.
($\Leftarrow$). Let the identity

$$f(x_1 \dots, x_n) = g(x_1 \dots, x_n)$$

hold on $(a]$, and on $[a)$. Then, for $y_1, \dots, y_n \in L$

$$f(y_1 \wedge a, \dots, y_n \wedge a) = g(y_1 \wedge a, \dots, y_n \wedge a), \text{ and}$$
$$f(y_1 \vee a, \dots, y_n \vee a) = g(y_1 \vee a, \dots, y_n \vee a).$$

By Lemma 1 and Lemma 2, it follows that

$$f(y_1, \dots, y_n) \wedge a = g(y_1, \dots, y_n) \wedge a, \text{ and}$$
$$f(y_1, \dots, y_n) \vee a = g(y_1, \dots, y_n) \vee a.$$

Now, by Lemma 3 it follows that

$$f(y_1, \dots, y_n) = g(y_1, \dots, y_n).$$

□

Lemma 6. *Let 1 be the greatest element of the lattice L, $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$ an arbitrary lattice identity, and $b \in L$.*
Now, if for all $x_1, \dots, x_{n-1} \in L$

$$f(x_1, \dots, x_{n-1}, 1) = g(x_1, \dots, x_{n-1}, 1),$$

then for all $x_1, \dots, x_{n-1} \in (b]$

$$f(x_1, \dots, x_{n-1}, b) = g(x_1, \dots, x_{n-1}, b).$$

Proof. Follows by the fact that 1 is the greatest element of L, and b the greatest in $(b]$. It is a straightforward proof by induction that any equality of the form

$$f(x_1,\dots,x_{n-1},b)=g(x_1,\dots,x_{n-1},b),$$

where $x_1,\dots,x_{n-1}\in(b]$, reduces to $b=b$, or to an equality not containing b.

□

Dually, we have the following.

Lemma 7. *Let 0 be the smallest element of the lattice L, $f(x_1,\dots,x_n)=g(x_1,\dots,x_n)$ an arbitrary lattice identity, and $b\in L$.*
If for all $x_1,\dots,x_{n-1}\in L$

$$f(x_1,\dots,x_{n-1},0)=g(x_1,\dots,x_{n-1},0),$$

then for all $x_1,\dots,x_{n-1}\in[b)$

$$f(x_1,\dots,x_{n-1},b)=g(x_1,\dots,x_{n-1},b).$$

□

Theorem 2. *Let a be a distributive, codistributive and comodular element of the lattice L, and let $f(x_1,\dots,x_n)=g(x_1,\dots,x_n)$ be an arbitrary lattice identity. Let also $b\in(a]$, and let $\bar{b}$ be the greatest element of the class to which b belongs, under $x\to x\wedge a$.*
Now, if the equality

$$f(x_1,\dots,x_{n-1},a)=g(x_1,\dots,x_{n-1},a)$$

is satisfied on $(a]$ and on $[a)$, then

$$f(x_1,\dots,x_{n-1},b)=g(x_1,\dots,x_{n-1},b)$$

holds on the ideal $(\bar{b}]$ (i.e. for all $x_1,\dots,x_{n-1}\in(\bar{b}]$).

Proof. Let $x_1,\dots,x_{n-1}\in(\bar{b}]$. Then, by Lemma 2 and Lemma 6

$$f(x_1,\dots,x_{n-1},b)\wedge a=f(x_1\wedge a,\dots,x_{n-1}\wedge a,b)=g(x_1\wedge a,\dots,x_{n-1}\wedge a,b)=$$

$$=g(x_1,\dots,x_{n-1},b)\wedge a,$$

since $x_1 \wedge a, \ldots, x_{n-1} \wedge a \in (b]$, and the corresponding equality holds on $(a]$ by assumption. Similarly, by Lemma 1,

$$f(x_1, \ldots, x_{n-1}, b) \vee a = f(x_1 \vee a, \ldots, x_{n-1} \vee a, b) = g(x_1 \vee a, \ldots, x_{n-1} \vee a, b) =$$
$$= g(x_1, \ldots, x_{n-1}, b) \vee a,$$

since the corresponding equality holds on the filter $[a)$.

Finally, by Lemma 3, it follows that for all $x_1, \ldots, x_{n-1} \in (\bar{b}]$

$$f(x_1, \ldots, x_{n-1}, b) = g(x_1, \ldots, x_{n-1}, b)$$

□

3. Application to algebra

Lemma 8. *The following are equivalent for an algebra $\mathcal{A}$:*

(0) $\mathcal{A}$ has the CEP;

(i) if for $\rho, \theta \in Con\mathcal{B}$, $\mathcal{B} \in Sub\mathcal{A}$, $\rho \vee \Delta = \theta \vee \Delta$ in $Cw\mathcal{A}$, then, $\rho = \theta$;

(ii) if for $\rho, \theta \in Cw\mathcal{A}$, $\rho \leq \theta$, then $\rho \vee (\Delta \wedge \theta^2) = (\rho \vee \Delta) \wedge \theta^2$;

(iii) if for $\rho \in Cw\mathcal{A}$ and $B \in Sub\mathcal{A}$, $\rho \leq B^2$, then $\rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2$;

(iv) for $\rho, \theta \in Cw\mathcal{A}$, $\rho \vee (\Delta \wedge \theta) = (\rho \vee \Delta) \wedge (\rho \vee \theta)$.

Proof. Algebraic reformulation of Lemma 3.

□

Proposition 1. *If an algebra $\mathcal{A}$ has the CIP, then $\mathcal{A}$ has the CEP if and only if for all $\rho, \theta \in Cw\mathcal{A}$, $\rho \wedge (\Delta \vee \theta) = (\rho \wedge \Delta) \vee (\rho \wedge \theta)$.*

Proof. By Lemma 4.

□

Proposition 2. *If an algebra $\mathcal{A}$ has the CEP, then $\mathcal{A}$ has the CIP if and only if for $\rho, \theta \in Cw\mathcal{A}$, $\theta \in Con\mathcal{B}$, $\mathcal{B} \in Sub\mathcal{A}$,*

$$\Delta_B \vee (\rho \wedge \theta) = (\Delta_B \vee \rho) \wedge \theta.$$

(Δ_B stands for $B^2 \cap \Delta$.)

Proof. By Lemma 5.

□

Theorem 3. *If an algebra $\mathcal{A}$ has the CEP and the CIP, then an arbitrary lattice identity is satisfied on $Cw\mathcal{A}$ if and only if this identity holds on $Sub\mathcal{A}$ and on $Con\mathcal{A}$.*

Proof. Immediately by Theorem 1, since Δ is a codistributive element of $Cw\mathcal{A}$, and since $\mathcal{A}$ has the CEP and the CIP if and only if Δ is distributive and comodular as well (the latter because of Lemma 8, (ii)).

□

The following two propositions were proved in [3]. We give them as the consequences of Theorem 3, and of the lattice interpretation of the CEP and the CIP (the preceding lemmas).

Corollary 1. *An algebra $\mathcal{A}$ has a modular lattice of weak congruences if and only if $\mathcal{A}$ has the CEP and the CIP, and $Sub\mathcal{A}$ and $Con\mathcal{A}$ are modular lattices.*

Corollary 2. *An algebra $\mathcal{A}$ has a distributive lattice of weak congruences if and only if $\mathcal{A}$ has the CEP and the CIP, and $Sub\mathcal{A}$ and $Con\mathcal{A}$ are distributive lattices.*

Theorem 4. *Let $\mathcal{A}$ be an algebra which has the CEP and the CIP, and let $f(x_1,\ldots,x_n) = g(x_1,\ldots,x_n)$ be an arbitrary lattice identity. If the equality*

$$f(x_1,\ldots,x_{n-1},A) = g(x_1,\ldots,x_{n-1},A)$$

is satisfied on $Sub\mathcal{A}$, and the equality

$$f(x_1,\ldots,x_{n-1},\Delta) = g(x_1,\ldots,x_{n-1},\Delta)$$

holds on $Con\mathcal{A}$, then for every subalgebra $\mathcal{B}$ of $\mathcal{A}$

$$f(x_1,\ldots,x_{n-1},\Delta_B) = g(x_1,\ldots,x_{n-1},\Delta_B)$$

holds on $Cw\mathcal{B}$.

Proof. By Theorem 2.

□

Corollary 3. *If an algebra $\mathcal{A}$ has the CEP and the CIP, then every subalgebra of $\mathcal{A}$ has the CIP.*

□

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REZIME

SPECIJALNI ELEMENTI MREŽE I MREŽNI IDENTITETI

Daju se uslovi pod kojima proizvoljan mrežni identitet važi na mreži, ako je on zadovoljen na podmrežama (nekim) te mreže, indukovanim specijalnim kongruencijama te mreže.

Navedene su algebarske primene tih rezultata na mrežama kongruencija, podalgebri i slabih kongruencija proizvoljne algebre.

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