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GROUP THEORETICAL METHODS IN GRAY-SCALE MATHEMATICAL MORPHOLOGY

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Abstract

We show that for the morphological operations involving a finite number of bounded functions a mathematical foundation in group-theoretical terms can be given which unifies the notions for subsets and for gray-scale images, and this as well in the Euclidean as in the digital case.

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1. Introduction

For some years, the operations of standard mathematical morphology for subsets of R^n or Z^n (dilation, erosion, opening, closing) have been generalized to the case of gray-scale images; we refer to [1] and [2] for a complete description of that method.

There seem to be, however, some draw-backs in that method, among which we mention:
(i) The use of the umbra transforms of the functions, instead of the functions themselves (which is more natural).

(ii) A separate construction has to be made for the morphology for subsets of R^2(or Z^2) and for gray-scale images.

It would be valuable if a uniform theory could be constructed in which no distinction has to be made either between the morphological operations for subsets and for gray-scale images, or between the Euclidean and the digital case, and without using the umbra transform.

In what follows we shall show that such a construction is indeed possible when we start by defining the morphological operations immediately for functions defined on subsets of a group G (e.g., G is $(R^2,+)$or$(Z^2,+)$) and with values in a suitable group V (V is either $(R,+)$ or $(Z,+)$ or $\{0\}$; for$V = \{0\}$ we obtain the notions for subsets). In that manner the construction of a dilation, ..., does not suffer from the draw-backs mentioned above.

In section 3 we shall define the r-dilation and r-erosion of two functions; the resulting quantities are in general not functions, but, rather, general subsets of $G \times V$. We shall then show in section 4 that, for bounded functions, the needed morphological operations dilation and erosion are obtained by taking the "surface" of the corresponding r-quantities in section 3. The other notions such as opening, closing, ... are then derived from these two. Of the multitude of properties only a few

are quoted at each step, just to give a flavour of them.

We finally mention that we only stressed the group theoretical facts; vector space or module concepts like convexity have not been considered here.

An algebraic basis for mathematical morphology in terms of complete lattices has also been constructed recently; for this we refer to [3] and [4].

2. Classical definitions of dilation and erosion

For selfcontainment, we state in this section the classical definitions of dilation and erosion, first for subsets of R^n, and then for functions defined on a subset of R^{n-1}; these definitions may be found in [1] and [2].

Given two subsets A and B of R^n, the dilation (or Minkowski addition) of A by B is the set $\mathcal{D}(A,B)$ defined by

$$\mathcal{D}(A,B) = \bigcup_{b \in B} (A+b),$$

while the erosion of A by B is the set $\mathcal{E}(A,B)$ defined by

$$\mathcal{E}(A,B) = \{x \in R^n : B + x \subset A\}$$

(see figure 1, figure 2).

For a gray-scale image, defined as a real-valued function f on a subset A of R^{n-1}, the umbra $U[f]$ of f is the set consisting of all the function values $f(x)$ and everything below it; this means $U[f] = \{(x,y) \in A \times R : y \leq f(x)\}$. Hence, the umbra of f is a subset of R^n. Given two real valued functions f and g defined on subsets A and B of R^{n-1} respectively, the dilation $\mathcal{D}(f,g)$ of f by g is defined as the "top" of the set-dilation of $U[f]$ and $U[g]$, i.e., for $x \in \mathcal{D}(A,B)$

$\mathcal{D}(f,g)(x) = \sup\{y : (x,y) \in \mathcal{D}(U[f],U[g])\}$, while the erosion $\mathcal{E}(f,g)$ of f by g is given as the top of the set-erosion of $U[f]$ and $U[g]$ (see figure 3, ..., figure 6).

3. Relational morphological operations

Let G be a (not necessarily commutative) group with a multiplicative group operation and group identity e. For $a \in G$ en $B \subset G$ we denote the left multiplication of B by a as aB, where $aB = \{ab : b \in B\}$; analogously, the right multiplication of B by a is $Ba = \{ba : b \in B\}$. The inverse of an element $a \in G$ is a^{-1}, and $B^{-1} = \{b^{-1} : b \in B\}$.

By V we denote either the additive group $(R, +)$ of real numbers with zero element 0, or the additive group $(Z, +)$ of integers again with zero 0, or the group $\{0\}$ containing only the zero 0.

$G \times V$ is also a group for the following operation: for $x \in G$, $y \in G$, $r \in V$, $s \in V$ we define

$$(x,r).(y,s) = (xy, r+s).$$

For $V = \{0\}, G \times V$ can be identified in the usual manner with G.

Let $f : D_f \to V$ an $g : D_g \to V$ be functions defined on subsets D_f and D_g of G. Clearly, any function like f is a subset of $G \times V$. We write $g << f$ when the following two conditions are satisfied:

(i) $D_g \subset D_f$

(ii) $g(y) \leq f(y)$ for all $y \in D_g$.

When $V = \{0\}$, the condition $g << f$ reduces to the ordinary inclusion relation $D_g \subset D_f$.

Definition 1. *The r-dilation* $\mathcal{D}r(f,g)$ *of two functions* $f : D_f \to V$ *and* $g : D_g \to V$ *is the subset of* $G \times V$ *defined by*

$$\begin{aligned} \mathcal{D}_r(f,g) &= \{(x, f(x)).(y, g(y)) : x \in D_f, y \in D_g\} \\ &= \{(xy, f(x) + g(y)) : x \in D_f, y \in D_g\}, \end{aligned}$$

Hence $\mathcal{D}_r(f,g)$ *is just another notation for the result of applying the group operation in* $G \times V$ *on its subsets* f *and* g.

For the special case when $V = \{0\}$, we identify f with D_f, g with $D_g, G \times V$ with G, and we obtain the usual definition of the dilation of two subsets D_f and D_g of G. From the definition it readily follows that for $V = \{0\}$ we have (with $h : D_h \to V$):

$$\mathcal{D}_r(D_f, D_g) = \bigcup_{y \in D_g} D_f y = \bigcup_{x \in D_f} x D_g,$$

$$\mathcal{D}_r(\mathcal{D}_r(D_f, D_g), D_h) = \mathcal{D}_r(D_f, \mathcal{D}_r(D_g, D_h)),$$

$$\text{for } f << h : \mathcal{D}_r(D_f, D_g) \subset \mathcal{D}_r(D_h, D_g).$$

We just give a proof for one property in case $V = \{0\}$.

Proposition 1.

$$\mathcal{D}_r(D_f, D_g) = \{z \in G : zD_g^{-1} \cap D_f \neq \emptyset\}.$$

Proof.

$$\begin{aligned} \mathcal{D}_r(D_f, D_g) &= \{z \in G : z \in U_{x \in D_f} x D_g\} \\ &= \{z \in G : \exists x \in D_f \text{ such that } x^{-1}z \in D_g\} \\ &= \{z \in G : \exists x \in D_f \text{ such that } x \in zD_g^{-1}\} \\ &= \{z \in G : zD_g^{-1} \cap D_f \neq \emptyset\}. \ \square \end{aligned}$$

For general V and $(z,t) \in G \times V$, the set $g.(z,t)$ is defined as the group product in $G \times V$ of the subset $\{(y, g(y)) : y \in D_g\}$ with the element (z,t); hence $g.(z,t) = \{(yz, g(y) + t) : y \in D_g\}$. We remark that $g.(z,t)$ is in fact just the right translate $g_{(z,t)}$ of the function g, defined by $g_{(z,t)}(u) = g(uz^{-1}) + t$ for all u such that $uz^{-1} \in D_g$, i.e.,

$u \in D_g z$. Analogously, the left translate ${}_{(z,t)}g$ of g is defined by ${}_{(z,t)}g(u) = g(z^{-1}u) + t$ for $u \in zD_g$, and this is precisely the subset $(z,t).g$. The following result then follows immediately

Proposition 2.

$$\mathcal{D}_r(f, g.(z,t)) = \mathcal{D}_r(f,g).(z,t)$$

$$\mathcal{D}_r((z,t).f, g) = (z,t).\mathcal{D}_r(f,g).\square$$

Definition 2. *The r-erosion* $\mathcal{E}_r(f,g)$ *of two functions* $f : D_f \to V$ *and* $g : D_g \to V$ *is the subset of* $G \times V$ *defined by*

$$\mathcal{E}_r(f,g) = \{(z,t) \in G \times V : (z,t).g << f\}.$$

Since

$$\begin{aligned}(z,t).g &= \{(zy, g(y) + t : y \in D_g\} \\ &= \{(u, g(z^{-1}u) + t : u \in zD_g\},\end{aligned}$$

we also obtain

$$\mathcal{E}_r(f,g) = \{(z,t) \in G \times V : zD_g \subset D_f, g(z^{-1}u) + t \le f(u)\}$$

for all u such that $z^{-1}u \in D_g$.

For fixed g it follows from $f << h$ that $\mathcal{E}_r(f,g) << \mathcal{E}_r(h,g)$, while for fixed f and $g << h$ we have that $\mathcal{E}_r(f,h) << \mathcal{E}_r(f,g)$. When $(y,s) \in G \times V$ it may also be verified that $\mathcal{E}_r(f,(y,s).g) = \mathcal{E}_r(f,g).(y,s)^{-1}$, and analogously $\mathcal{E}_r((y,s).f, g) = (y,s).\mathcal{E}_r(f,g)$.

For $V = \{0\}$ we obtain

$$\begin{aligned}\mathcal{E}_r(D_f, D_g) &= \{z \in G : zD_g \subset D_f\} \\ &= \bigcap_{y \in D_g} \{z \in G : zy \in D_f\} \\ &= \bigcap_{y \in D_g} D_f y^{-1}.\end{aligned}$$

Moreover, when $e \in D_g$ then $\mathcal{E}_r(D_f, D_g) \subset D_f$.

4. True gray-scale morphological operations

The relational morphologic operations of section 3 associate with two gray-scale images f and g a new subset of $G \times V$ which, except for the trivial case when $V = \{0\}$, is in general not a function, and hence is not a new image. In order to obtain from the r-dilation and r-erosion real gray-scale images (which will then be called the dilation and erosion), it is sufficient to take a suitable subset of those relational morphologic structures. This is easily obtained by taking the surface of such a subset, according to the following definition.

Definition 3. *When $\mathcal{A} \subset G \times V$, we call surface of $\mathcal{A}$ (denoted by $S(\mathcal{A})$) the set*

$$S(\mathcal{A}) = \{(x,t) \in G \times V : x \in proj_G \mathcal{A}, t = \sup\{s \in V : (x,s) \in \mathcal{A}\}\},$$

where $proj_G \mathcal{A}$ denotes the projection of $\mathcal{A}$ on G, and where sup possibly might be replaced by maximum; we always suppose that sup exists in V. Hence, $S(\mathcal{A})$ is a function defined on $proj_G \mathcal{A}$.

Proposition 3. *Let $\mathcal{A} \subset G \times V, (z,t) \in G \times V$. Then*

$$S(\mathcal{A}.(z,t)) = S(\mathcal{A}).(z,t); S((z,t).\mathcal{A}) = (z,t).S(\mathcal{A}).$$

Proof. We shall just prove the first equality. Since $(z,t) = (z,0).(e,t)$, it is sufficient to prove that $S(\mathcal{A}.(e,t)) = S(\mathcal{A}).(e,t)$ and that $S(\mathcal{A}.(z,0)) = S(\mathcal{A}).(z,0)$.

That $S(\mathcal{A}.(e,t)) = S(\mathcal{A}).(e,t)$ follows from the fact that $\mathcal{A}.(e,t) = \{(u,s+t) : (u,s) \in \mathcal{A}\}$, which leads to $proj_G(\mathcal{A}.(e,t)) = proj_G \mathcal{A}$, and from the fact that $S(\mathcal{A}.(e,t)) = \{(x,r) \in G \times V : x \in proj_G \mathcal{A}, r = \sup\{s+t : (x,s) \in \mathcal{A}\}\}$, and hence $r = \sup\{s : (x,s) \in \mathcal{A}\} + t$. In an analogous manner we obtain $proj_G(\mathcal{A}.(z,0)) = (proj_G \mathcal{A})z$, and when $(x,s) \in S(\mathcal{A}.(z,0))$, then $s = \sup\{r \in V : (x,r) \in \mathcal{A}\}$. □

Now we are able to introduce the gray-scale morphological notions for functions $f : D_f \to V$ and $g : D_g \to V$, which we suppose to be bounded.

Definition 4. *The dilation $\mathcal{D}(f,g)$ of f and g is defined by*

$$\mathcal{D}(f,g) \quad = \quad S(\mathcal{D}_r(f,g))$$

$$
\begin{aligned}
&= \{(z,t) \in G \times V : z \in proj_G \mathcal{D}_r(f,g), t \\
&= \sup\{s \in V : (z,s) \in \mathcal{D}_r(f,g)\}\}.
\end{aligned}
$$

In view of the definition of $\mathcal{D}_r(f,g), z$ belongs to $D_f D_g$, and hence it can be written in different manners in the form $z = xy$ with $x \in D_f, y \in D_g$; the value of $\mathcal{D}(f,g)$ in such a point z is then given by

$$
\begin{aligned}
\mathcal{D}(f,g)(z) &= \sup\{f(x) + g(x^{-1}z) : x \in D_f, x^{-1}z \in D_g\} \\
&= \sup\{f(zy^{-1}) + g(y) : y \in D_g, zy^{-1} \in D_f\}.
\end{aligned}
$$

According to our definition, dilation of two bounded functions is an operation which is always defined and is associative. For $V = \{0\}$ it leads to the usual definition of dilation of two subsets, and in that case $\mathcal{D}(f,g)$ and $\mathcal{D}_r(f,g)$ coincide.

In view of the results given before, a lot of properties about dilation may be proved. We shall just mention a few of them and give only a proof of the first one.

Proposition 4. *For* $(z,r) \in G \times V$ *we have*

$$\mathcal{D}(f, g.(z,r)) = \mathcal{D}(f,g).(z,r); \mathcal{D}((z,r).f, g) = (z,r).\mathcal{D}(f,g).$$

Proof.

$$
\begin{aligned}
\mathcal{D}(f, g.(z,r)) &= S(\mathcal{D}_r(f, g.(z,r))) = S(\mathcal{D}_r(f,g).(z,r)) = \\
&= S(\mathcal{D}_r(f,g)).(z,r) = \mathcal{D}(f,g).(z,r). \quad \square
\end{aligned}
$$

Proposition 5. *If* $f_1 << f_2$ *then* $\mathcal{D}(f_1,g) << \mathcal{D}(f_2,g)$. □

Proposition 6. *If* $D_g = D_h$, *then* $\mathcal{D}(f, \max(g,h)) = \max(\mathcal{D}(f,g), \mathcal{D}(f,h))$.
□

Definition 5. *The erosion* $\mathcal{E}(f,g)$ *of* f *and* g *is defined by*

$$
\begin{aligned}
\mathcal{E}(f,g) &= S(\mathcal{E}_r(f,g)) \\
&= \{(x,t) \in G \times V : xD_g \subset D_f, t \\
&= \sup\{s \in V : g(x^{-1}u) + s \leq f(u)\}\}
\end{aligned}
$$

for all u *such that* $x^{-1}u \in D_g$.

Hence the domain of $\mathcal{E}(f,g)$ is the set of those points $x \in G$ such that $xD_g \subset D_f$; the value t of $\mathcal{E}(f,g)(x)$ may also be given by one of the following expressions:

$$\begin{aligned}\mathcal{E}(f,g)(x) &= \sup_{y\in D_g}\{s \in V : g(y) + s \leq f(xy)\}\\ &= \sup_{y\in D_g}\{s \in V : s \leq f(xy) - g(y)\}\\ &= \inf_{y\in D_g}\{f(xy) - g(y)\}.\end{aligned}$$

For $V = \{0\}, \mathcal{E}(f,g)$ and $\mathcal{E}_r(f,g)$ coincide, giving the usual definition of erosion of two subsets.

Proposition 7. *For* $(z,r) \in G \times V$ *we have*

$$\mathcal{E}(f,(b,r).g) = \mathcal{E}(f,g).(b,r)^{-1}$$

Proof.

$$\begin{aligned}\mathcal{E}(f,(b,r).g) &= S(\mathcal{E}_r(f(b,r).g) = S(\mathcal{E}_r(f,g).(b,r)^{-1}) =\\ &= \mathcal{E}(f,g).(b,r)^{-1}.\square\end{aligned}$$

Proposition 8. *For* $f_1 << f_2$ *we have* $\mathcal{E}(f_1,g) << \mathcal{E}(f_2,g)$. $\square$

Proposition 9. *For* $g_1 << g_2$ *we have* $\mathcal{E}(f,g_2) << \mathcal{E}(f,g_1)$. $\square$

The other morphologic notions can be derived from the two basic ones, dilation and erosion. We restrict ourselves to the definition of opening.

Definition 6. *The opening* $0(f,g)$ *of two functions* f *and* g *is defined by*

$$0(f,g) = \mathcal{D}(\mathcal{E}(f,g),g);$$

hence,

$$0(f,g)(z) = \sup\{\mathcal{E}(f,g)(x) + g(x^{-1}z) : x \in D_{\mathcal{E}(f,g)}, x^{-1}z \in D_g\},$$

which leads to one of the following alternative expressions

$$\begin{aligned}0(f,g)(z) &= \sup\{\inf_{y\in D_g}\{f(xy) - g(y)\} + g(x^{-1}z) : xD_g \subset D_f, z \in xD_g\}\\ &= \sup\{\sup_{y\in D_g}\{t : t \leq f(xy) - g(y)\} +\\ &\quad g(x^{-1}z) : xD_g \subset D_f, z \in xD_g\}.\end{aligned}$$

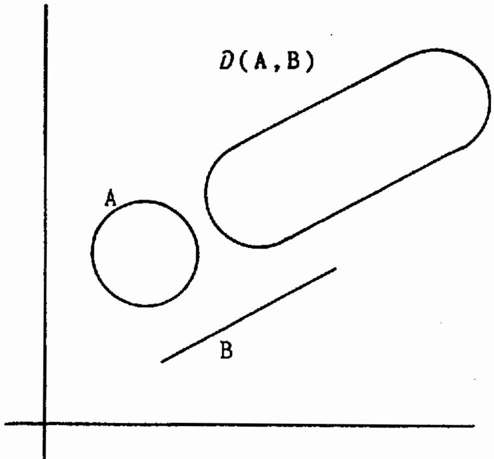

figure 1

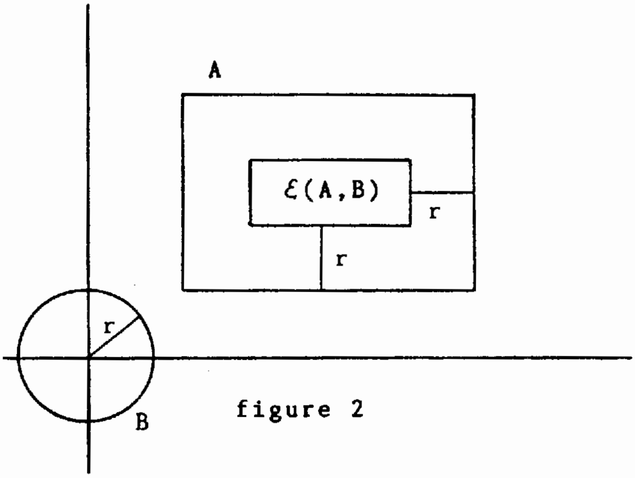

figure 2

It will be clear from the foregoing that, based on the notion of functions defined on some subset of a group and with values in a suitable subgroup of R, all the necessary notions pertaining to the morphology of subsets or grayscale images, whether digital or not, may be derived in a unifying manner using only the functions themselves.

In any case, it is no longer necessary to make a distinction in the definition between the cases where subsets are used, or functions defined on subsets; nor do we have to distinguish between continuous or digital images. One definition is sufficient, since we only have to adapt the set V. Morever, the umbra transform has not been used. This leads to a very economic way of treating the morphologic machinery.

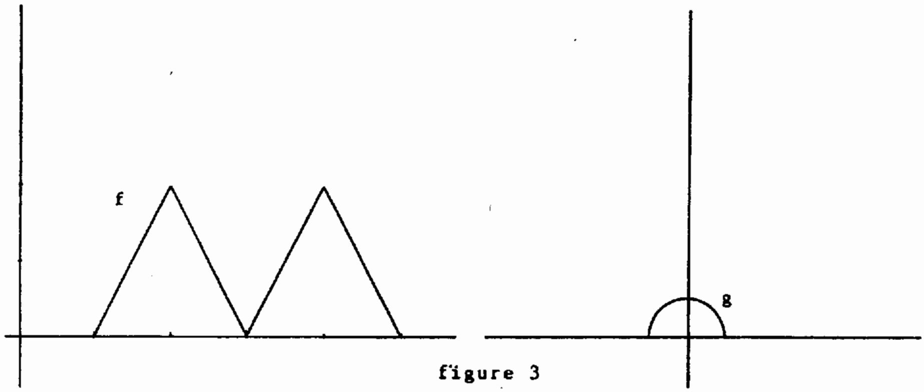

figure 3

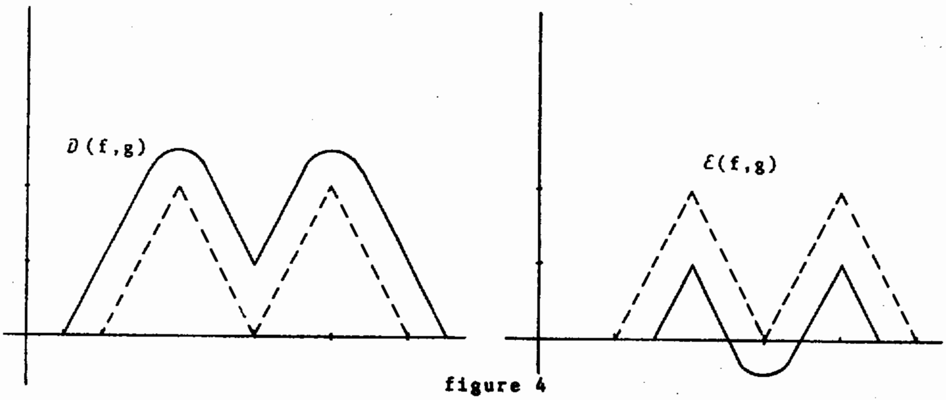

figure 4

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REZIME

GRUPOVNO TEORETSKE METODE U MATEMATIČKOJ MORFOLOGIJI SIVIH NIVOA

Pokazano je da za morfološke operacije koje uključuju konačan broj ograničenih funkcija, može biti dato matematičko zasnivanje u grupovno teoretskim terminima, što unificira pojmove podskupova i sivih nivoa i to i u Euklidovom i digitalnom slučaju.

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