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SOME RESULTS ON THE REDUCED ENERGY OF GRAPHS, I

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Abstract

In a recent paper [6], A.Torgašev described all the finite connected graphs whose energy (i. e. the sum of all the positive eigenvalues including also their multiplicities) does not exceed 3. In this paper are described all the connected graphs whose reduced energy, i. e. the sum of absolute values of all the eigenvalues except the largest one, does not exceed 5.

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In this paper we shall consider only finite connected graphs having no loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and its order (number of vertices) by $|G|$. The spectrum of such a graph is the set $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of eigenvalues of its $0-1$ adjacency matrix.

The sum of eigenvalues $|\lambda_2| + \cdots + |\lambda_n|$ is denoted by $R_1(G)$ and called the **reduced** energy of G. We note that $|\lambda_n| \geq 1$, hence $R_1(G) \geq 1$ for any graph G. For any real $a \geq 1$, we can consider the class of graphs

$$C_1(a) = \{G|\ R_1(G) \leq a\}.$$

In this paper we describe completely the class $C_1(5)$.

Briefly, any graph $G \in C_1(5)$ is called **admissible**, and any other graph - **impossible** (or forbidden) for this class.

If H is any connected (induced) subgraph of a graph G **we denote it** by $H \subseteq G$. Making use of the known interlacing theorem [1;p.19] we have

$R_1(H) \leq R_1(G)$, whence we have that any connected subgraph of an admissible graph is also admissible. This implies that the method of forbidden subgraphs can be consistently applied.

We shall firstly prove an important property of the general class

$$C_1(a)\ (a \geq 1).$$

Theorem 1. *The class $C_1(a)$ is finite for every $a \geq 1$.*

Proof. Let G be an arbitrary graph from the class $C_1(a)$. Then we have

$$a \geq \sum_{i=2}^{n} |\lambda_i| \geq \sum_{\lambda_i<0} |\lambda_i| = \sum_{\lambda_i>0} |\lambda_i|,$$

thus $G \in S(a)$, where

$$S(a) = \{G| \sum_{\lambda_i>0} |\lambda_i| \leq a\}$$

is the class treated in paper [6]. Hence, $C_1(a) \subseteq S(a)$. Since by Theorem 2 in [6] , the class $S(a)$ is finite for every $a \geq 1$, our Theorem is proved.

□

Next, let $K_{n_1 n_2 \ldots n_m}$, P_n and C_n be the complete m–partite graph, the path and the cycle with n vertices, respectively. Since the complete m–partite graph $K_{n_1 n_2 \ldots n_m}$ has just one positive eigenvalue $r(G)$, it will belong to the class $C_1(a)$ if and only if $r(G) \leq a$.

We shall firstly determine the exact values of parameters $n_1, n_2, \ldots, n_m$ for which the graph $K_{n_1 n_2 \ldots n_m}(n_1 \leq n_2 \leq \ldots \leq n_m)$ is admissible.

Proposition 1. *The graph $K_{n,m}$ $(n \leq m)$ is admissible exactly for the following values of parameters n,m:*

1. $n = 1,\ m = 1, 2, \ldots, 25.$

2. $n = 2,\ m = 2, 3, \ldots, 12.$

3. $n = 3,\ m = 3, 4, \ldots, 8.$

4. $n = 4,\ m = 4, 5, 6.$

5. $n = 5,\ m = 5.$

Proof. Since the graph $K_{n,m}$ is the complete bipartite graph, we have that $r(G) = \sqrt{nm}$. Therefore $G \in C_1(5)$ if and only if $nm \leq 25$, which easily gives the statement.

□

Proposition 2. *The graph $K_{n,m,k}(n \leq m \leq k)$ is admissible exactly for the following values of parameters n,m,k:*

1. $n = 1,\ m = 1,\ k = 1, 2, \dots, 10.$
2. $n = 1,\ m = 2,\ k = 2, 3, \dots, 6.$
3. $n = 1,\ m = 3,\ k = 3, 4.$
4. $n = 2,\ m = 2,\ k = 2, 3.$

Proof. The characteristic polynomial of this graph is

$$P(\lambda) = \lambda^{n+m+k-3}(\lambda^3 - (nm + nk + mk)\lambda - 2nmk).$$

Hence, it is a matter of routine to see that $r(G) \leq 5$ exactly for the mentioned values of parameters n, m, k.

□

Proposition 3. *The graph $K_{n,m,k,l}(n \leq m \leq k \leq l)$ is admissible exactly for the following values of parameters n, m, k, l:*

$$(n,m,k,l) = (1,1,1,1),(1,1,1,2),(1,1,1,3),(1,1,1,4),\\ (1,1,1,5),(1,1,2,2),(1,1,2,3).$$

Proof. It is easy to check that all the above graphs are admissible. Besides, since the graph $K_{n,m,k,l}$ is forbidden for the values of parameters $(n,m,k,l) = (1,1,1.6),(1,1,2,4),(1,1,3,3)$, the interlacing theorem provides the statement.

□

Proposition 4. *The graph $K_{n,m,k,l,p}(n \leq m \leq k \leq l \leq p)$ is admissible exactly for the following values of parameters n, m, k, l, p:*

$$(n,m,k,l,p) = (1,1,1,1,1),(1,1,1,1,2)$$

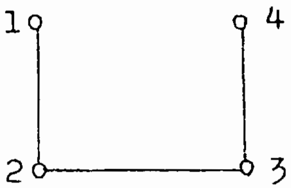
Proposition 5. *The graph* $G = K_{n_1 n_2 \dots n_m} (m \geq 6)$ *is admissible if and only if* $m = 6$ *and* G *is the graph* K_6.

Proof. The graph K_6 is obviosly an admissible graph. Besides, since the graphs $K_{1,1,1,1,1,2}$ and K_7 are forbidden, the statement is immediate.

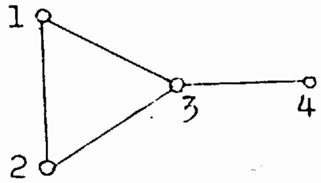
□

Next, we shall determine all the admissible graphs from the class $C_1(5)$ wich have at least two positive eigenvalues. By a result of Smith [5] we have

Theorem A *Graph* G *has at least two positive eigenvalues if and only if it has one of the following graphs as an (induced) subgraph.*



G_1



G_2

Note that the mentioned graphs G_1 and G_2 are admissible. Denote $\mathcal{G}_1 = \{G_1, G_2\}$. If $G \in \mathcal{G}_1$ and S is any nonempty subset from the vertex set $V(G)$, let G_x be the graph obtained from G by adding a new vertex x adjacent just to the vertices from S. Denote by $\mathcal{G}_2$ the set of all the nonisomorphic admissible graphs G_x, where $G \in \mathcal{G}_1$, and $S \subseteq V(G) \setminus \{\emptyset\}$. If the class $\mathcal{G}_i$ is constructed, we define $\mathcal{G}_{i+1}$ by the class obtained by $\mathcal{G}_i$ in a similar way as $\mathcal{G}_2$ has been obtained by $\mathcal{G}_1$. Denote

$$\mathcal{G} = \cup_{i \in N} \mathcal{G}_i$$

By theorem 1, class $\mathcal{G}$ is finite, i.e. we have $\mathcal{G}_i = \emptyset$ for all i large enough. All the graphs $G \in \mathcal{G}$ are, by definition admissible graphs, and $\mathcal{G}$ contains all such graphs with more than one positive eigenvalue. This follows from the fact that for any connected graph G and any of its connected induced subgraph H, there is a sequence of connected induced subgraphs $H_i \subseteq G$ $(i = 0, 1, \dots, r)$ such that

$$H = H_O \subseteq H_1 \subseteq \dots \subseteq H_r = G$$

and

$$|H_{i+1}| = |H_i| + 1 \quad (i = 0, 1, \ldots, r-1).$$

Using a computer device, we can generate the set of all the nonisomorphic admissible graphs with at least two positive eigenvalues.

In this way we get the following theorem.

Theorem 2. *The class $C_1(5)$ contains exactly 137 nonisomorphic graphs which have at least two positive eigenvalues. The orders of these graphs run over the set $\{4, 5, \ldots, 11\}$. All these graphs are represented in List 1.*

Proposition 1 – 5 and Theorem 3 describe completely the class $C_1(5)$. In particular, we get that class $C_1(5)$ contains exactly $75 + 137 = 212$ nonisomorphic graphs.

We should notice that all the graphs in this list are represented in the form

$$n_1 \; n_2 \; n_3 \;\; a_{12} \; a_{13} \; a_{23} \ldots a_{1n} \; a_{2n} \ldots a_{n-1,n},$$

where n_1 is the ordering number of a corresponding graph, n_2 is the number of its vertices, n_3 is the number of its edges and $a_{12}\; a_{13}\; a_{23} \ldots a_{1n}\; a_{2n} \ldots a_{n-1,n}$ is the upper diagonal part of its adjacency matrix.

LIST 1. ADMISSIBLE GRAPHS WITH AT LEAST TWO POSITIVE EIGENVALUES

001 04 03 1 10 001
002 04 04 1 10 110

003 05 04 1 10 001 1000
004 05 04 1 10 001 0100
005 05 05 1 10 001 0101
006 05 05 1 10 011 1000
007 05 05 1 10 001 1100
008 05 05 1 10 110 1000
009 05 05 1 10 001 1010
010 05 06 1 10 001 1110
011 05 06 1 10 110 1001
012 05 06 1 10 001 1101
013 05 06 1 10 110 1010
014 05 07 1 11 111 1000
015 05 07 1 10 011 1101
016 05 07 1 10 111 0011
017 05 08 1 11 111 1001

018 06 05 1 10 001 1000 10000
019 06 05 1 10 001 1000 00010
020 06 05 1 10 001 1000 01000
021 06 05 1 10 001 1000 00100
022 06 06 1 10 001 1010 00100
023 06 06 1 10 001 0100 01100
024 06 06 1 10 001 1010 00001
025 06 06 1 10 011 1000 00100
026 06 06 1 10 011 1000 10000
027 06 06 1 10 011 1000 00001
028 06 06 1 10 110 1000 10000
029 06 06 1 10 110 1000 00100
030 06 06 1 10 001 1000 00110
031 06 07 1 10 001 1000 11100
032 06 07 1 10 001 1000 10110
033 06 07 1 10 001 1010 10100

034 06 07 1 10 110 1001 00100
035 06 07 1 10 110 1001 10000
036 06 07 1 10 100 1000 01101
037 06 07 1 10 001 1101 01000
038 06 07 1 10 001 1000 01101
039 06 08 1 10 100 1111 00010
040 06 08 1 10 100 1111 00001
041 06 08 1 10 111 0011 00010
042 06 08 1 10 011 1101 00001
043 06 08 1 10 011 1101 00100
044 06 08 1 10 110 1001 00110
045 06 08 1 10 011 1001 00011
046 06 08 1 10 011 1001 00101
047 06 08 1 10 110 1000 11010
048 06 08 1 10 001 1010 10101
049 06 09 1 11 111 1001 10000
050 06 09 1 11 111 1001 01000
051 06 09 1 10 110 1000 01111
052 06 09 1 10 011 1000 01111
053 06 09 1 10 110 1000 10111
054 06 09 1 10 011 1101 01001
055 06 09 1 10 001 1101 01110
056 06 09 1 10 011 1000 11110
057 06 09 1 10 110 1001 11001
058 06 10 1 10 111 0111 00011
059 06 10 1 10 111 0111 10001
060 06 10 1 11 111 1001 10010
061 06 10 1 10 110 1001 11011
062 06 10 1 10 011 1001 01111
063 06 11 1 11 111 1000 01111
064 06 11 1 10 011 1101 11110
065 06 11 1 10 111 1111 00011
066 06 12 1 10 111 0111 11011

067 07 06 1 10 100 1000 10000 000001
068 07 06 1 10 001 1000 00100 001000
069 07 06 1 10 001 1000 00010 100000
070 07 06 1 10 001 1000 00010 000100

071	07	07	1	10	011	1000	10000	000100	
072	07	07	1	10	001	1000	10000	101000	
073	07	07	1	10	100	1000	10000	010010	
074	07	07	1	10	110	1000	10000	100000	
075	07	07	1	10	001	1000	00100	101000	
076	07	08	1	10	001	1000	01101	001000	
077	07	08	1	10	100	1000	10000	001101	
078	07	08	1	10	110	1001	10000	000100	
079	07	08	1	10	100	1000	10000	100011	
080	07	08	1	10	011	1000	00001	100100	
081	07	08	1	10	100	1000	01101	000001	
082	07	09	1	10	100	1000	01101	100001	
083	07	09	1	10	001	1010	00100	011010	
084	07	09	1	10	100	1000	10000	101110	
085	07	09	1	10	110	1000	11010	100000	
086	07	09	1	10	001	1000	10000	011011	
087	07	09	1	10	001	1010	10100	101000	
088	07	10	1	10	100	1000	01111	100010	
089	07	10	1	10	011	1001	01101	001000	
090	07	10	1	10	100	1000	01111	001100	
091	07	10	1	10	100	1111	00001	100001	
092	07	10	1	10	110	1001	11001	100000	
093	07	10	1	10	100	1000	11111	000001	
094	07	10	1	10	011	1101	00001	100100	
095	07	11	1	10	011	1001	01111	000001	
096	07	11	1	10	011	1001	00101	011010	
097	07	11	1	10	110	1000	10000	011111	
098	07	11	1	10	111	0111	10001	000100	
099	07	12	1	10	100	1111	00001	011110	
100	07	12	1	10	110	1001	10000	011111	
101	07	13	1	10	100	1111	00001	111101	
102	07	13	1	10	100	1000	01111	011111	
103	07	13	1	10	110	1000	01111	011110	
104	07	14	1	10	111	0111	10001	011101	
105	07	14	1	10	111	0111	11101	011000	
106	08	07	1	10	001	1000	00100	001000	1000000
107	08	07	1	10	001	1000	00100	001000	0010000
108	08	07	1	10	100	1000	10000	100000	0000010

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109 08 07 1 10 001 1000 00010 100000 1000000
110 08 08 1 10 100 1000 10000 100000 0000101
111 08 08 1 10 001 1000 00100 101000 1000000
112 08 08 1 10 110 1000 10000 100000 1000000
113 08 08 1 10 110 1000 10000 100000 0100000
114 08 09 1 10 100 1000 10000 100000 0010101
115 08 09 1 10 100 1000 10000 100000 1100010
116 08 09 1 10 110 1001 10000 000100 0001000
117 08 09 1 10 001 1000 10000 101000 1010000
118 08 10 1 10 100 1000 10000 100000 1011100
119 08 10 1 10 100 1000 01101 100001 1000000
120 08 10 1 10 001 1010 10100 101000 1000000
121 08 10 1 10 100 1000 01101 100001 0000010
122 08 11 1 10 100 1000 10000 100000 0101111
123 08 11 1 10 100 1000 10000 100000 1101110
124 08 11 1 10 100 1000 11111 000001 1000000
125 08 11 1 10 100 1000 01111 100001 0000001
126 08 12 1 10 100 1000 10000 100000 1011111
127 08 13 1 10 110 1000 10000 100000 0111111
128 08 16 1 10 100 1000 01111 100001 0111111

129 09 08 1 10 100 1000 10000 100000 1000000 01000000
130 09 09 1 10 110 1000 10000 100000 1000000 10000000
131 09 10 1 10 100 1000 10000 100000 1100010 10000000
132 09 13 1 10 100 1000 10000 100000 0111111 00000001
133 09 14 1 10 100 1000 10000 100000 1000000 11101111

134 10 09 1 10 100 1000 10000 100000 1000000 01000000 100000000
135 10 10 1 10 100 1000 10000 100000 1000000 10000000 100000001

136 11 10 1 10 100 1000 10000 100000 1000000 01000000 100000000 1000000000
137 11 11 1 10 100 1000 10000 100000 1000000 10000000 100000001 1000000000
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REZIME

NEKI REZULTATI O PRVOJ REDUKOVANOJ ENERGIJI GRAFA

U nedavno objavljenom radu [6], A. Torgašev je opisao sve konačne povezane grafove čija energija (tj. suma svih pozitivnih sopstvenih vrednosti uključujući takodje njihove višestrukosti) nije veća od 3. U ovom radu opisujemo sve povezane grafove čija prva redukovana energija, tj. suma apsolutnih vrednosti svih sopstvenih vrednosti bez maksimalne, nije veća od 5.

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