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ON A CONSTRUCTION OF CODES BY P–FUZZY SETS

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Abstract

P–fuzzy sets are considered as mappings from an arbitrary nonempty set S into a partially ordered set P. The necessary and sufficient conditions are given under which a family P of subsets of S represents a collection of level subsets for a fuzzy set $\overline{A} : S \to P$. Thus the conditions are obtained under which a binary block-code V can be ordered, so that it uniquely determines a P–fuzzy set and vice-versa. An explicit description of a Hamming distance for such codes is given, and it is shown that some well known binary block-codes (BCD, Gray's codes) can be represented by P–fuzzy sets.

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1. P–fuzzy sets

Let S be an arbitrary set which is not empty, and $(P, \leq)$ a partially ordered set. Any function $\overline{A} : S \to P$ is a *P–fuzzy set on S*. Let also for $p \in P$, $\overline{A_p} : S \to \{0, 1\}$, so that for $x \in S$, $\overline{A_p}(x) = 1$ iff $\overline{A}(x) \geq p$. Obviously, $\overline{A_p}$ is a characteristic function of a p–level subset (or, a p–cut)

$$A_p = \{x \in S | \ \overline{A_p}(x) = 1\}.$$

Let $\overline{A} : S \to P$ be a P–fuzzy set on S, and $\sim$ a binary relation on P, such that for $p, q \in P$

$$p \sim q \text{ iff } A_p = A_q.$$

$\sim$ is obviously an equivalence relation on P. Let

$$F = \overline{A}(S) = \{p \in P | p = \overline{A}(x), \text{ for some } x \in S\},$$

and for $p \in P$, let

$$[p) = \{q \in P | p \le q\}.$$

Lemma 1. *If $\overline{A} : S \to P$ is a P–fuzzy set on S, then for $p, q \in P$*

$$p \sim q \text{ iff } [p) \cap F = [q) \cap F.$$

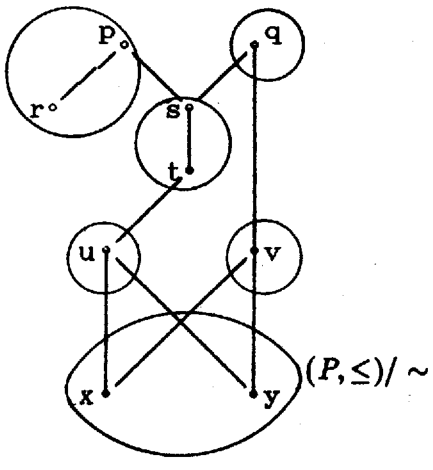
Proof.

$$\begin{aligned} p \sim q \quad &\text{iff} \quad A_p = A_q \text{ iff (for } x \in S)(\overline{A}(x) \ge p \text{ iff } \overline{A}(x) \ge q) \\ &\text{iff} \quad \{x \in S | \overline{A}(x) \in [p)\} = \{x \in S | \overline{A}(x) \in [q)\} \\ &\text{iff} \quad [p) \cap F = [q) \cap F. \quad \square \end{aligned}$$

Example 1.

$$S = \{a, b, c, d, e\} \quad P = \{p, q, r, s, t, u, v, x, y\}$$

$$\overline{A} = \begin{pmatrix} a & b & c & d & e \\ s & u & v & p & q \end{pmatrix}$$



	a	b	c	d	e
	s	u	v	p	q
$\overline{A_p}$	0	0	0	1	0
$\overline{A_q}$	0	0	0	0	1
$\overline{A_r}$	0	0	0	1	0
$\overline{A_s}$	1	0	0	1	1
$\overline{A_t}$	1	0	0	1	1
$\overline{A_u}$	1	1	0	1	1
$\overline{A_v}$	0	0	1	0	1
$\overline{A_x}$	1	1	1	1	1
$\overline{A_y}$	1	1	1	1	1

$$A_p = A_r = \{d\};\ \ A_q = \{e\};\ \ A_s = A_t = \{a,d,e\};$$

$$A_u = \{a,b,d,e\};\ \ A_v = \{c,e\};\ \ A_x = A_y = \{a,b,c,d,e\}$$

Lemma 2. *Let $\overline{A}: S \to P$ be a fuzzy set. Now for every $x \in S$, if $\overline{A}(x) = p$, then p is a supremum of the class to which it belongs, i.e. $p = \bigvee[p]_\sim$.*

Proof. If $q \in [p]_\sim$, then $p = \overline{A}(x) \geq q$. Hence, $p = \bigvee[p]_\sim$. □

The following statement is a Theorem of decomposition for P–fuzzy sets.

Theorem 1. *If $\overline{A}: S \to P$ is a P–fuzzy set on S, then for $x \in S$,*

$$\overline{A}(x) = \bigvee(p \in P|\overline{A_p}(x) = 1)$$

(i.e. the supremum on the right exists in $(P, \leq)$ for every $x \in S$, and is equal to $\overline{A}(x)$).

Proof. Let $\overline{A}(x) = r \in P$. Then, $\overline{A_r}(x) = 1$. Now, if for any $p \in P$ $\overline{A_p}(x) = 1$, then $\overline{A}(x) \geq p$, i.e. $r \geq p$. On the other hand, $r \in \{p \in P|\overline{A_p}(x) = 1\}$, and thus r is the greatest element of that family. Thus,

$$\overline{A}(x) = r = \bigvee(p|\overline{A_p}(x) = 1). \quad \square$$

Let $\overline{A_P} = \{A_p | p \in P\}$, for $\overline{A}: S \to P$.This family of subsets of S has the following properties:

Proposition 1. *For a P–fuzzy set $\overline{A}: S \to P$,*

(1) if $p, q \in P$ and $p \leq q$, then $A_q \subseteq A_p$;

(2) if for $P_1 \subseteq P$ there exists a supremum of P_1 ($\bigvee(p|p \in P_1)$), then $\bigcap(A_p|p \in P_1) = A_{\bigvee(p|p\in P_1)}$;

(3) $\bigcup(A_p|p \in P) = S$;

(4) for every $x \in S$, $\bigcap(A_p|x \in A_p) \in \overline{A_P}$.

Proof.

(1) If $p \leq q$, then $\overline{A_q}(x) = 1$ implies $\overline{A_p}(x) = 1$, i.e. $A_q \subseteq A_p$;

(2) Suppose that for $P_1 \subseteq P$ the supremum $\bigvee(p|p \in P_1)$ exists in P. Then for $x \in S$,

$$\begin{aligned} x \in A_{\bigvee(p|p\in P_1)} \quad &\text{iff} \quad \overline{A}_{\bigvee(p|p\in P_1)}(x) = 1 \ \text{ iff } \ \overline{A}(x) \geq \bigvee(p|p \in P_1) \\ &\text{iff} \quad \overline{A}(x) \geq p \ \text{ for all } \ p \in P_1, \\ &\text{iff} \quad x \in \bigcap(A_p|p \in P_1)\,; \end{aligned}$$

(3) If $x \in S$, then $\overline{A}(x) = p \in P$ and $x \in A_p$. Thus, $x \in \bigcup(A_p|p \in P)$, i.e. $S \subseteq \bigcup(A_p|p \in P)$. Obviously, $\bigcup(A_p|p \in P) \subseteq S$, and the equality holds;

(4) Let $x \in S$. Then, $x \in A_p$ iff $\overline{A}(x) \geq p$, i.e. iff $\overline{A_p}(x) = 1$. By Theorem 1, $\overline{A}(x) = \bigvee(p|\overline{A_p}(x) = 1)$, and by (2)

$$A_{\bigvee(p|\overline{A_p}(x)=1)} = \bigcap(A_p|\overline{A_p}(x) = 1).$$

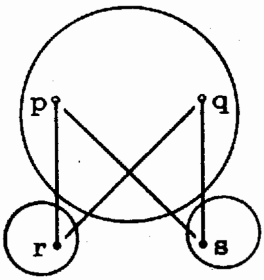
Hence, $\bigcap(A_p|x \in A_p) \in \overline{A_P}$. □

Remark. The converse of (2) in Proposition 1 is not true, as shown by the following example.

Example 2.

$$S = \{a, b\} \quad P = \{p, q, r, s\}$$

$$\overline{A} = \begin{pmatrix} a & b \\ r & s \end{pmatrix}$$



$(P, \leq)/\sim$

	a	b
	r	s
$\overline{A_p}$	0	0
$\overline{A_q}$	0	0
$\overline{A_r}$	1	0
$\overline{A_s}$	0	1

$$A_p = A_q = \emptyset$$
$$A_r = \{a\}$$
$$A_s = \{b\}$$

In this P–fuzzy set, $A_r \cap A_s \in \overline{A_P}$, but (2), (Proposition 1) is not true, since $r \vee s$ does not exist in P.

Theorem 2. *Let S be a nonempty set, and P a family of its subsets ($P \subseteq \mathcal{P}(S)$), such that:*

(1) $\bigcup P = S$;

(2) for every $x \in S$, $\bigcap(p \in P | x \in p) \in P$.

Let $\overline{A} : S \to P$ *be defined with*

$$\overline{A}(x) = \bigcap(p \in P | x \in p).$$

Then, $\overline{A}$ *is a* P*–fuzzy set, where* $(P, \leq)$ *is a partially ordered set under* $p \leq q$ *iff* $q \subseteq p$ $(p, q \in P)$, *and for every* $p \in P$,

$$p = A_p.$$

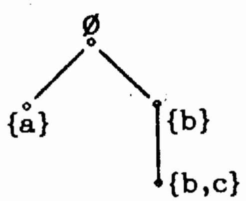
Proof. $\overline{A}$ is well defined. Indeed, by (2), for every $x \in S$ the family $\{p \in P | x \in p\}$ is uniquely determined.

$\overline{A}$ is obviously a P–fuzzy set, and we have to prove that for every $p \in P$, $p = A_p$ (recall that $A_p = \{x \in S | \overline{A}(x) \geq p\}$).

Let $x \in S$. Then,
$x \in A_p$ iff $\overline{A}(x) \geq p$ iff (by the definition of $\overline{A}$)
$\bigcap(q \in P | x \in q) \geq p$ iff (by the definition of $\leq$)
$\bigcap(q \in P | x \in q) \subseteq p$ iff $x \in p$ (since by (1), the intersection is not empty).
□

Example 3. $S = \{a, b, c\}$ $P = \{\emptyset, \{a\}, \{b\}, \{b, c\}\}$
Conditions (1) and (2) are satisfied. Thus, we have the following P*–fuzzy set:*

$$\overline{A} = \begin{pmatrix} a & b & c \\ \{a\} & \{b\} & \{b,c\} \end{pmatrix}$$



$(P, \leq)$
$(p \leq q$ *iff* $q \subseteq p)$

	a	b	c	
$\overline{A_p}$	$\{a\}$	$\{b\}$	$\{b,c\}$	A_p
$A_\emptyset$	0	0	0	$\emptyset$
$A_{\{a\}}$	1	0	0	$\{a\}$
$A_{\{b\}}$	0	1	0	$\{b\}$
$A_{\{b,c\}}$	0	1	1	$\{b,c\}$

As shown in the table, for every $p \in P$, $A_p = p$.

2. Codes generated by P–fuzzy sets

Let $S = \{1, 2, ...n\}$ and let $(P, \leq)$ be a finite partially ordered set. Every P–fuzzy set on S *determines* a binary block-code V of length n, in the following way:

To every class $[p]_\sim$ $(p \in P)$, there corresponds a codeword

$v_{[p]} = x_1 x_2 \ldots x_n$, such that $x_i = j$ iff $\overline{A_p}(i) = j$, for $i \in S$, and $j \in \{0, 1\}$.

We shall use the following componentwise defined order on the set of codewords belonging to a binary block-code V: for $x, y \in V$, $x = x_1 ... x_n$, $y = y_1 ... y_n$,

$$(*) \qquad x \leq y \quad \text{iff} \quad y_1 \leq x_1, ..., y_n \leq x_n,$$

where $\leq$ on the right is the ordinary ordering relation on the lattice $(\{0, 1\}, \leq) : 0 < 1$.

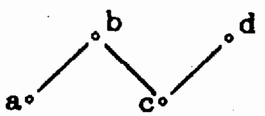
Thus, for example, $101101 \leq 001001$.

Theorem 3. *Every finite partially ordered set* $(P, \leq)$ *determines a block-code* V, *such that* $(P, \leq)$ *is isomorphic with* $(V, \leq)$.

Proof. Let $P = \{p_1, ...p_n\}$, and let $\overline{A} : P \to P$ be the identity mapping, as a P–fuzzy set on P. The decomposition of $\overline{A}$ gives a family $\{\overline{A_p} | p \in P\}$ which is the required code, under the above defined order $(*)$. Consider the mapping $f : P \to \{\overline{A_p} | p \in P\}$, such that $f(p) = \overline{A_p}$. By Lemma 2 every $(\sim)$-class contains exactly one element, and thus f is one-to-one. If $p, q \in P$ and $p \leq q$, then $A_q \subseteq A_p$, which by $(*)$ means that $\overline{A_p} \leq \overline{A_q}$, and f is an isomorphism. □

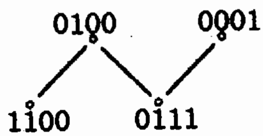
Example 4. $P = \{a, b, c, d\}$

$$\overline{A} = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} \qquad V = \{1100, 0100, 0111, 0001\}$$



$(P, \leq)$

$\overline{A_x}$	a	b	c	d
$\overline{A_a}$	1	1	0	0
$\overline{A_b}$	0	1	0	0
$\overline{A_c}$	0	1	1	1
$\overline{A_d}$	0	0	0	1



$(V, \leq)$

If V is a binary block-code, and $\overline{A}$ is a P–fuzzy set, then we say that $\overline{A}$ *corresponds* to V if the block-code determined by $\overline{A}$ (as defined at the beginning of the paragraph) is V.

Theorem 4. *Let $V = \{v_1, ...v_k\} \subseteq \{0,1\}^n$ be a binary block-code, such that for every $i \in \{1,\dots n\}$ at least one codeword has a nonzero i–th coordinate. Then there is a P–fuzzy set which corresponds to V if and only if for every $i \in \{1,\dots n\}$*

$$(a) \qquad \bigvee(v \in V | v(i) = 1) \in V.$$

(The supremum in (a) is induced by $\leq$ in $(*)$.)
Proof. If we think of the codewords of V as of the characteristic functions of the subsets of $\{1, ...n\}$, then the "if" part follows by Theorem 2, since the supremum in (a) is in fact the intersection of the corresponding subsets.

The converse is true by Proposition 1. □

Recall that the *Hamming weight* $||x||$ of a codeword $x \in \{0,1\}^n$ is the number of the nonzero coordinates in x, the *Hamming distance* $d(x,y)$ between x and y from $\{0,1\}^n$ is the number of coordinates in which x and y differ. The *code distance* of $V \subseteq \{0,1\}^n$ is the minimum Hamming distance between two different codewords in V, and is denoted by $d(V)$.

Let $\overline{A} : S \to P$ be a P–fuzzy set. We say that the number of elements of S which are mapped into the same element p of P is a *degree* of the class $[p]_\sim$, or of the corresponding codeword $v_{[p]}$, and we denote it by $s(v_{[p]})$.

In the following four propositions, let V be a code corresponding to a P–fuzzy set $\overline{A} : S \to P$.

As it was done in [3] for lattice valued fuzzy sets, we shall describe the above-mentioned parameters in terms of P–fuzzy sets.

Proposition 2.

$$d(V) \geq min_{p \in \overline{A}(S)} s(v_{[p]}).$$

(Recall that $\overline{A}(S) = \{p \in P | p = \overline{A}(x), \quad \text{for some } x \in S\}$.)
Proof. If two codewords from V differ in the coordinate mapped onto p, they differ in at least $s(v_{[p]})$ coordinates. □

Proposition 3. *If $v_{[p]} \in V$, then*

$$||v_{[p]}|| = \sum(s(v_{[q]}) | q \in \overline{A}(S), \ \ \textit{and} \ \ v_{[p]} \leq v_{[q]}).$$

Proof. If $i \in S$, and $v_{[p]} \in V$, then $v_{[p]}(i) = 1$ if $q = \overline{A}(i) \geq r$, for every $r \in [p]_{\sim}$. Every $q \in \overline{A}(S)$ represents one class $[q]_{\sim}$ (by Lemma 2), and the number of these classes coincides with $||v_{[p]}||$. □

Proposition 4. *If* $v_{[p]} \leq v_{[q]}$, *then*

$$d(v_{[p]}, v_{[q]}) = \sum(s(v_{[r]})|v_{[r]} \in K),$$

$$\textit{where} \quad K = \{v_{[r]} \in V | v_{[r]} \geq v_{[p]}, \textit{ and } \rceil(v_{[r]} \geq v_{[q]})\}.$$

Proof. If $v_{[r]} \geq v_{[p]}$, and $\rceil(v_{[r]} \geq v_{[q]})$, $r \in \overline{A}(S)$, then for every $i \in S$ such that $\overline{A}(i) = r$, $v_{[p]}(i) = 1$, and $v_{[q]}(i) = 0$. Moreover, every nonzero coordinate in $v_{[q]}$ is nonzero in $v_{[p]}$ as well. □

Theorem 5. *For any* $v_{[p]}, v_{[q]} \in V$,

$$(b) \qquad d(v_{[p]}, v_{[q]}) = \sum(s(v_{[r]})|v_{[r]} \in K)$$

where

$$K = \{v_{[r]} \in V | v_{[r]} \geq v_{[p]} \textit{ and } \rceil(v_{[r]} \geq v_{[q]}), \textit{ or } v_{[r]} \geq v_{[q]} \textit{ and } \rceil(v_{[r]} \geq v_{[p]})\}.$$

Proof. If $v_{[p]} \leq v_{[q]}$, then the proof follows by Proposition 4. Now,let $v_{[p]}$ and $v_{[q]}$ be uncomparable. If $v_{[r]} \geq v_{[p]}$, and $\rceil(v_{[r]} \geq v_{[q]})$, then for every $i \in S$ such that $\overline{A}(i) = r$, $v_{[p]}(i) = 1$ and $v_{[q]}(i) = 0$. On the other hand, if $v_{[r]} \geq v_{[q]}$, $\rceil(v_{[r]} \geq v_{[p]})$ and $\overline{A}(i) = r$, then $v_{[q]}(i) = 1$, and $v_{[p]}(i) = 0$. Hence,

$$d(v_{[p]}, v_{[q]}) \geq \sum(s(v_{[r]})|v_{[r]} \in K).$$

Now, if $v_{[p]}$ and $v_{[q]}$ differ in i–th coordinate, for example $v_{[q]}(i) = 1$ and $v_{[p]}(i) = 0$, then for $\overline{A}(i) = r$, $v_{[r]} \geq v_{[q]}$ and $\rceil(v_{[r]} \geq v_{[p]})$. Thus, $v_{[r]} \in K$, and the equality (b) holds. □

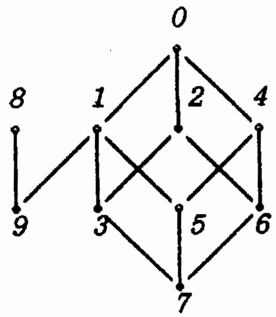
* * *

Some well known codes can be represented by P–fuzzy sets, i.e. they satisfy the conditions of Theorem 4.

Example 5. *a) For a BCD-code* $V = \{0000, 0001, \ldots 1001\}$, *there is a corresponding* $P-$*fuzzy set, as shown in the sequel.*

$$\overline{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 4 & 2 & 1 \end{pmatrix},$$

$$S = \{1, 2, 3, 4\}, \ P = \{1, 2, \ldots 9\}$$



$(P, \leq)$

	1	2	3	4
p	8	4	2	1
0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1

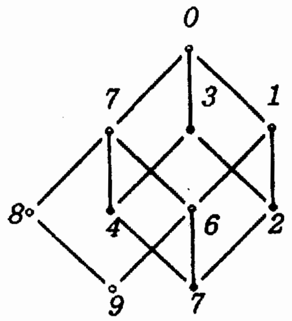
The construction is based on Theorem 4.

b) The following Gray's code can be represented by a $P-$fuzzy set in a similar way.

$V = \{0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101\}$

$S = \{1, 2, 3, 4\}$ $\qquad \overline{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 3 & 1 \end{pmatrix}$

$P = \{0, 1, \ldots 9\}$



$(P, \leq)$

	1	2	3	4
p	8	7	3	1
0	0	0	0	0
1	0	0	0	1
2	0	0	1	1
3	0	0	1	0
4	0	1	1	0
5	0	1	1	1
6	0	1	0	1
7	0	1	0	0
8	1	1	0	0
9	1	1	0	1

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REZIME

O KONSTRUKCIJI KODOVA POMOĆU $P-$RASPLINUTIH SKUPOVA

Posmatraju se $P-$rasplinuti skupovi, kao funkcije iz proizvoljnog nepraznog skupa S u parcijalno uredjeni skup P. Daju se potrebni i dovoljni uslovi pod kojima familija podskupova skupa S predstavlja kolekciju na koju se razlaže dati rasplinuti skup na S. Time se dolazi do uslova pod kojima se binarni blok-kod može opisati pomoću $P-$rasplinutog skupa. Eksplicitno se opisuje norma, Hemingovo i uopšte kodno rastojanje, a daju se i primeri poznatih kodova (BCD, kodovi Greja) izraženih pomoću $P-$rasplinutih skupova.

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