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THE PRODUCT OF THE DISTRIBUTIONS x_+^λ AND $x_-^{-\lambda-p}$ DEFINED BY A DISTRIBUTION VECTOR

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Abstract

This paper proves the existence of the neutrix product $x_+^\lambda \circ x_-^{-\lambda-p}$ of the distributions x_+^λ and $x_-^{-\lambda-p}$ in terms of the distribution vector which was defined in [3].

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In the following N denotes the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions $n^\lambda \ln^{r-1} n$, $\ln^r x$ for $\lambda > 0$ and $r = 1, 2, \ldots$ and all functions which converge to zero in the normal sense as n tends to infinity.

Now let ρ be a fixed infinitely differentiable function having the properties

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

The function δ_n is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$. It is obvious that the sequence $\{\delta_n\}$ is regular and converges to the Dirac delta-function δ.

The following definition for the product of two distributions was given in [2].

Definition 1. *Let f and g be distributions and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the interval (a, b) if*

$$\operatorname*{N-lim}_{n\to\infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle$$

for all test functions ϕ with compact support contained in the interval (a, b).

In order to give more information about the behaviour of the product $f \circ g$ the notion of a distribution vector was introduced in [3], where the following definitions, theorems and lemma were given.

Definition 2. *Let h_r be a distribution for $r = 0, 1, 2, \ldots$. We say that*

$$\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$$

is a distribution vector.

If α is any real number we define $\alpha\mathbf{h}$ to be the distribution vector

$$[\alpha h_0, \alpha h_1, \ldots, \alpha h_r, \ldots]$$

and if $\mathbf{k} = [k_0, k_1, \ldots, k_r, \ldots]$ is a second distribution vector we define $\mathbf{h} + \mathbf{k}$ to be the distribution vector

$$[h_0 + k_0, h_1 + k_1, \ldots, h_r + k_r, \ldots].$$

If $h_{r+i} = 0$ for $i = 1, 2, \ldots$ we write

$$\mathbf{h} = [h_0, h_1, \ldots, h_r, 0, 0, \ldots] = [h_0, h_1, \ldots, h_r]$$

and if $h_i = 0$ for $i = 1, 2, \ldots$ we write

$$\mathbf{h} = [h_0] = h_0.$$

Definition 3. *Let* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *be a distribution vector and let* ϕ *be an arbitrary test function with compact support. We define* $\langle \mathbf{h}, \phi \rangle$ *by the sequence of real numbers*

$$\langle \mathbf{h}, \phi \rangle = (\langle h_0, \phi \rangle, \langle h_1, \phi \rangle, \ldots, \langle h_r, \phi \rangle, \ldots).$$

Definition 4. *Let* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *be a distribution vector. We define the derivative* $\mathbf{h}'$ *of* $\mathbf{h}$ *by*

$$\mathbf{h}' = [h_0', h_1', \ldots, h_r', \ldots].$$

Definition 5. *Let* f *and* g *be distributions and let* $g_n = g * \delta_n$. *We say that the neutrix product* $f \circ g$ *of* f *and* g *exists and is equal to the distribution vector* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *on the interval* (a, b) *if*

$$\operatorname*{N-lim}_{n \to \infty} n^{-r} \langle f g_n, \phi \rangle = \langle h_r, \phi \rangle$$

for $r = 0, 1, 2, \ldots$ *and all test functions* ϕ *with compact support contained in the interval* (a, b).

Definition 6. *Let* f *and* g *be distributions and suppose that the neutrix product* $f \circ g$ *exists as the distribution vector* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *on the interval* (a, b). *We say that* h_0 *is the finite part of* $f \circ g$ *on the interval* (a, b). *If* $h_r \neq 0$ *for some* $r \geq 1$ *we write*

$$\text{p.f. } f \circ g = h_0$$

on the interval (a, b) *and if* $h_r = 0$ *for* $r = 1, 2, \ldots$ *we write*

$$f \circ g = h_0$$

on the interval (a, b).

Theorem 1. *Let* $\mathbf{h} = [h_0, h_1, \ldots, h_r, \ldots]$ *be a distribution vector and let* ϕ *be an arbitrary test function with compact support. Then*

$$\langle \mathbf{h}', \phi \rangle = -\langle \mathbf{h}, \phi' \rangle.$$

Theorem 2. *Let f and g be distributions and suppose that the neutrix products $f \circ g$ and $f' \circ g$ (or $f \circ g'$) exist as distribution vectors on the interval (a,b). Then the neutrix product $f \circ g'$ (or $f' \circ g$) exists as a distribution vector and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on the interval (a,b).

Lemma 1.

$$\int_0^{1/n} x^p \delta_n^{(q)}(x)\, dx = (-1)^{p+1} n^{q-p} p! \rho_{q-p-1}$$

for $p = 0,1,2,\ldots,q-1$ and $q = 1,2,\ldots$,

$$\int_0^{1/n} x^p \delta_n^{(p)}(x) dx = \frac{1}{2}(-1)^p p!$$

for $p = 0,1,2,\ldots$ and

$$\int_0^{1/n} \left| x^p \delta_n^{(q)}(x) \right| dx = o(n^{q-p})$$

for $q = 0,1,\ldots,p-1$ and $p = 1,2,\ldots$, where

$$\rho_{2r} = \rho^{(2r)}(0), \quad \rho_{2r+1} = 0$$

for $r = 0,1,2,\ldots$.

We now prove the following theorem.

Theorem 3. *The neutrix product $x_+^\lambda \circ x_-^{-\lambda-q}$ exists as a distribution vector and*

(1) $$x_+^\lambda \circ x_-^{-\lambda-q} = \mathbf{h}(\lambda,q) = [h_0(\lambda,q), h_1(\lambda,q), \ldots, h_{q-1}(\lambda,q)]$$

for $q = 1,2,\ldots$ and $\lambda \neq 0, \pm 1, \pm 2, \ldots$, where

$$h_i(\lambda,q) = \begin{cases} -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(q-1)!}\delta^{(q-1)}, & i = 0, \\ \frac{\Gamma(\lambda+q-i)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)(q-i-1)!}\rho_{i-1}\delta^{(q-i-1)}, & 1 \leq i < q. \end{cases}$$

In particular

$$x_+^\lambda \circ x_-^{-\lambda-1} = -\frac{1}{2}\pi\operatorname{cosec}(\pi\lambda)\delta$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. The distribution x_+ is defined as a locally summable function for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0 \end{cases}$$

and is defined for $\lambda < -1$ and $\lambda \neq -2, -3, \dots$ inductively by

$$x_+^\lambda = (\lambda+1)^{-1}(x_+^{\lambda+1})'.$$

The distribution x_-^λ is defined by

$$x_-^\lambda = (-x)_+^\lambda$$

for $\lambda \neq -1, -2, \dots$.

We will first of all suppose that $p - 1 < \lambda < p$, for some non-negative integer p. Then

$$x_-^{-\lambda-q} = \frac{\Gamma(\lambda-p+1)}{\Gamma(\lambda+q)}\frac{d^{p+q-1}}{dx^{p+q-1}}x_-^{-\lambda+p-1}$$

and so

$$\begin{aligned} (x_-^{-\lambda-q})_n &= x_-^{-\lambda-q} * \delta_n(x) \\ &= \frac{\Gamma(\lambda-p+1)}{\Gamma(\lambda+q)}\int_x^{1/n}(t-x)^{-\lambda+p-1}\delta_n^{(p+q-1)}(t)dt \end{aligned}$$

for $-1/n < x < 1/n$. Thus

$$\frac{\Gamma(\lambda+q)}{\Gamma(\lambda-p+1)}\int_{-\infty}^{\infty} x_+^\lambda (x_-^{-\lambda-q})_n x^j dx =$$

$$= \int_0^{1/n} x^{\lambda+j}\int_x^{1/n}(t-x)^{-\lambda+p-1}\delta_n^{(p+q-1)}(t)\,dt\,dx =$$

$$= \int_0^{1/n}\delta_n^{(p+q-1)}(t)\int_0^t x^{\lambda+j}(t-x)^{-\lambda+p-1}\,dx\,dt =$$

$$= \int_0^{1/n} t^{p+j}\delta_n^{(p+q-1)}(t)\int_0^1 v^{\lambda+j}(1-v)^{-\lambda+p-1}\,dv\,dt,$$

where the substitution $x = tv$ has been made. Since

$$\int_0^1 v^{\lambda+j}(1-v)^{-\lambda+p-1}\,dv = \frac{\Gamma(\lambda+j+1)\Gamma(-\lambda+p)}{(p+q)!},$$

it follows on using the lemma that

$$\int_{-\infty}^{\infty} x_+ (x_-^{-\lambda-q})_n x^j\,dx = \tag{2}$$

$$= \begin{cases} \frac{(-1)^j \Gamma(\lambda+j+1)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)} \rho_{q-j-2} n^{q-j-1}, & 0 \le j \le q-2 \\ \frac{1}{2}(-1)^q \pi\operatorname{cosec}(\pi\lambda), & j = q-1 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \left| x_+^\lambda (x_-^{-\lambda-q})_n x^q \right| dx = o(n^{-1}). \tag{3}$$

Now let ϕ be an arbitrary test function with compact support. Then

$$\phi(x) = \sum_{j=0}^{q-1} \frac{x^j}{j!} \phi^{(j)}(0) + \frac{x^q}{q!} \phi^{(q)}(\xi x),$$

where $0 < \xi < 1$. It follows that

$$\begin{aligned} \langle x_+^\lambda, (x_-^{-\lambda-q})_n \phi \rangle &= \sum_{j=0}^{q-1} \frac{\phi^{(j)}(0)}{j!} \int_{-\infty}^{\infty} x_+^\lambda (x_-^{-\lambda-q})_n x^j\,dx + \\ &\quad + \frac{1}{q!} \int_{-\infty}^{\infty} x_+^\lambda (x_-^{-\lambda-q})_n x^q \phi^{(q)}(\xi x)\,dx = \\ &= \sum_{j=0}^{q-2} \frac{\Gamma(\lambda+j+1)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q) j!} \rho_{q-j-2} n^{q-j-1} \langle \delta^{(j)}, \phi \rangle + \\ &\quad + \frac{\pi\operatorname{cosec}(\pi\lambda)}{2(q-1)!} \langle \delta^{(q-1)}, \phi \rangle + o(n^{-1}) \end{aligned}$$

on using equations (2) and (3). Thus

$$\mathrm{N}\!-\!\lim_{n\to\infty} n^{-i} \langle x_+^\lambda, (x_-^{-\lambda-q})_n \phi \rangle =$$

$$= \begin{cases} -\frac{\pi\operatorname{cosec}(\pi\lambda)}{2(q-1)!} \langle \delta^{(q-1)}, \phi \rangle, & i = 0 \\ \frac{\Gamma(\lambda+q-i)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)(q-i-1)!} \rho_{i-1} \delta^{(q-i-1)}, & 1 \le i \le q-1 \end{cases}$$

and equation (1) follows for $p-1 < \lambda < p$ and $p = 0, 1, 2, \dots$.

Now assume that equation (1) holds for $p-1 < \lambda < p$ and $q = 1, 2, \dots$, where p is now some negative integer. Then the neutrix products $x_+^\lambda \circ x_-^{-\lambda-q}$

and $x_+^\lambda \circ x_-^{-\lambda-q-1}$ exist and so by Theorem 2 the neutrix product $x_+^{\lambda-1} \circ x_-^{-\lambda-q}$ exists and

$$\begin{aligned}
\lambda x_+^{\lambda-1} \circ x_-^{-\lambda-q} &= (x_+^\lambda \circ x_-^{-\lambda-q})' - (\lambda+q)x_+ \circ x_-^{-\lambda-q-1} \\
&= \mathbf{h}'(\lambda,q) - (\lambda+q)\mathbf{h}(\lambda,q+1) \\
&= [k_0(\lambda,q), k_1(\lambda,q), \ldots, h_q(\lambda,q)],
\end{aligned}$$

where

$$\begin{aligned}
k_0(\lambda,q) &= h_0'(\lambda,q) - (\lambda+q)h_0(\lambda,q+1) \\
&= \frac{\pi\lambda\operatorname{cosec}(\pi\lambda)}{2q!}\delta^{(q)} \\
&= \lambda h_0(\lambda-1,q+1), \\
k_i(\lambda,q) &= h_i'(\lambda,q) - (\lambda+q)h_i(\lambda,q+1) \\
&= -\frac{\lambda\Gamma(\lambda+q-i)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)(q-i)!}\rho_{i-1}\delta^{(q-i)} \\
&= \lambda h_i(\lambda-1,q+1),
\end{aligned}$$

for $i = 1,2,\ldots,q-1$ and

$$\begin{aligned}
k_q(\lambda,q) &= -\frac{\lambda\Gamma(\lambda)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)}\rho_q\delta \\
&= \lambda h_q(\lambda-1,q+1).
\end{aligned}$$

Equation (1) follows for $p-2<\lambda<p-1$ and $q=1,2,\ldots$. Equation (1) now holds by induction for $p-1<\lambda<p$, $q=1,2,\ldots$ and $p=-1,-2,\ldots$. This completes the proof of the theorem. □

Corollary 1. *The neutrix product $x_-^\lambda \circ x_+^{-\lambda-q}$ exists as a distribution vector and*

$$x_-^\lambda \circ x_+^{-\lambda-q} = (-1)^{q-1}\mathbf{h}(\lambda,q)$$

for $q=1,2,\ldots$ and $\lambda \neq 0,\pm1,\pm2,\ldots$.

In particular

$$x_-^\lambda \circ x_+^{-\lambda-1} = -\frac{1}{2}\pi\operatorname{cosec}(\pi\lambda)\delta$$

for $\lambda \neq 0,\pm1,\pm2,\ldots$.

Proof. The results follow from the theorem on noting that $\rho_i = 0$ for odd i and

$$(-x)_+^\lambda = x_-^\lambda, \quad (-x)_-^{-\lambda-q} = x_+^{-\lambda-q},$$
$$\delta^{(i)}(-x) = (-1)^i \delta^{(i)}(x). \quad \square$$

Corollary 2.

$$\text{p.f. } x_+^\lambda \circ x_-^{-\lambda-q} = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(q-1)!}\delta^{(q-1)},$$
$$\text{p.f. } x_-^\lambda \circ x_+^{-\lambda-q} = \frac{(-1)^q \pi \operatorname{cosec}(\pi\lambda)}{2(q-1)!}\delta^{(q-1)}$$

for $q = 1, 2, \ldots$ *and* $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

The results of this corollary are immediate.

Corollary 3.

$$(\lambda+1)x_+^\lambda \circ x_-^{-\lambda-2} - (-1)^q(\lambda+q+1)x_+^{\lambda+q} \circ x_-^{-\lambda-q-2} = \tfrac{1}{2} q\pi \operatorname{cosec}(\pi\lambda)\delta'$$

for $q = \pm 1, \pm 2, \ldots$ *and* $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. The result follows on noting that

$$x_+^\lambda \circ x_-^{-\lambda-2} = \frac{\pi \operatorname{cosec}(\pi\lambda)}{\lambda+1}[-\frac{1}{2}(\lambda+1)\delta', \rho_0\delta],$$
$$x_+^{\lambda+q} \circ x_-^{-\lambda-q-2} = \frac{(-1)^q \pi \operatorname{cosec}(\pi\lambda)}{\lambda+q+1}[-\frac{1}{2}(\lambda+q+1)\delta', \rho_0\delta]. \quad \square$$

Corollary 4.

$$\pi x_+^{q-1} \circ \delta^{(q)} - (\lambda+q+1)(q-1)! \sin(\pi\lambda) x_+^{\lambda+q} \circ x_-^{-\lambda-q-2} = \frac{1}{2}(-1)^q(\lambda+1)(q-1)!\delta'$$

for $q = 1, 2, \ldots$ *and* $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. It was proved in [3] that

$$x_+^{q-1} \circ \delta^{(q)} = (-1)^q (q-1)! [-\frac{1}{2} q\delta', \rho_0\delta]$$

and the result follows on noting that

$$(\lambda+q+1)(q-1)! \sin(\pi\lambda) x_+^{\lambda+q} \circ x_-^{-\lambda-q-2} = (-1)^q \pi (q-1)! [\frac{1}{2}(\lambda+q+1)\delta', \rho_0\delta].$$

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REZIME

PROIZVOD DISTRIBUCIJA x_+^λ I $x_-^{-\lambda-p}$ DEFINISAN PREKO DISTRIBUCIONOG VEKTORA

Koristeći pojam distribucionog vektora iz [3], ispituje se proizvod distribucija x_+^λ i $x_-^{-\lambda-p}$, gde je λ realan broj koji nije ceo, a p prirodan broj.

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