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R^n-VALUED FUZZY RANDOM VARIABLE

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Abstract

A new metric and structure in the space of integrably bounded fuzzy random variables is introduced and the notion of conditional expectation for fuzzy random variable in R^n is defined.

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1. Introduction

The concept of a fuzzy set was introduced by Zadeh (1965). Subsequent developments focused on applications of this concept to pattern recognition and system analysis among other areas. Puri and Ralescu [9] studied fuzzy random variables as a generalization of random sets. The purpose of this generalization was the introduction of statistical techniques. The notion of conditional expectation is one of steps in that direction.

After some preliminaries on random sets introduced in § 2, we introduce a new distance on the set of integrably bounded fuzzy random variables in § 3. The main purpose of this paper is to present a theory of conditional expectation for fuzzy random variable and it is done in § 4.

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2. Preliminaries

In this paper we restrict our attention to the set of fuzzy random variables on the base space R^n, adapting in what follows definitions and results from Feron [3] and Puri, Ralescu [9]. A fuzzy set $u \in \mathcal{F}(R^n)$ is a function $u : R^n \to [0,1]$ for which

1. $u_0 = \overline{co}\{x \in R^n; u(x) > 0\}$ is compact,
2. the α-level set u_α of u, defined by

$$u_\alpha = \{x \in R^n : u(x) \geq \alpha\}$$

is nonempty, closed and convex subset of R^n for all $\alpha \in (0,1]$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space where P is a probability measure. A fuzzy random variable is a function $X : \Omega \to \mathcal{F}(R^n)$ such that

$$\{(\omega, x) : x \in (X(\omega))_\alpha\} \in \mathcal{A} \times \mathcal{B}, \text{ for every } \quad \alpha \in [0,1],$$

where $\mathcal{B}$ denotes the Borel subsets of R^n.

It is obvious that the function $X_\alpha : \Omega \to 2^{R^n}$ defined by $X_\alpha(\omega) = (X(\omega))_\alpha$ is the R^n-valued random set. If H is Hausdorff metric defined on $\mathcal{P}(R^n)$ (the space of all compact and convex subsets of R^n)

$$H(A,B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\}, \quad A, B \in \mathcal{P}(R^n),$$

then $(\mathcal{P}(R^n), H)$ is a complete metric space.

For any multufunction $F : \Omega \to \mathcal{P}(R^n)$ we can define the set

$$S_F = \{f \in L(\Omega, \mathcal{A}) : f(\omega) \in F(\omega) \quad P - a.e.\}$$

where $L(\Omega, \mathcal{A}) = L$ denotes the set of all functions $h : \Omega \to R^n$ which are integrable with respect to the probability measure P.

The set $S_F \subset L$ is closed with respect to a norm in L defined by

$$\|h\| = \int_\Omega \|h(\omega)\| dP, \quad h \in L.$$

Using S_F we can now define an integral for F (first introduced by Aumann [1])

$$\int_\Omega F dP = \{\int_\Omega f(\omega)dP(\omega) : f \in S_F\}.$$

The integrals $\int_\Omega f(\omega)dP(\omega)$ are defined in the sense of Bochner. $F : \Omega \to \mathcal{P}(R^n)$ is integrably bounded if there exists integrable real valued function $h : \Omega \to R$ such that $\sup_{x\in F(\omega)} \|x\| \le h(\omega)$ $P - a.e.$ The fuzzy random variable $X : \Omega \to \mathcal{F}(R^n)$ is integrably bounded if X_α is integrably bounded for all $\alpha \in [0,1]$. Let $\mathcal{L} = \mathcal{L}(\Omega, \mathcal{A})$ denote the set of all integrably bounded multivalued functions $F : \Omega \to \mathcal{P}(R^n)$ and let $\Lambda = \Lambda(\Omega, \mathcal{A})$ be the set of all integrably bounded fuzzy random variables $X : \Omega \to \mathcal{F}(R^n)$.

We shall close this section, by recalling a lemma which we shall use in the sequel.

Lemma 1. *([7]) Let M be a set and let $\{M_\alpha : \alpha \in [0,1]\}$ be a family of subsets of M such that*

1. $M_0 = M$
2. $\alpha \le \beta \Rightarrow M_\beta \subseteq M_\alpha$
3. $\alpha_1 \le \alpha_2 \le \ldots, \lim_{n\to\infty} \alpha_n = \alpha \Rightarrow M_\alpha = \bigcap_{n=1}^\infty M_{\alpha_n}$.

Then, the function $\phi : M \to [0,1]$ defined by $\phi(x) = sup\{\alpha \in [0,1] : x \in M_\alpha\}$ has the property that $\{x \in M : \phi(x) \ge \alpha\} = M_\alpha$ for every $\alpha \in [0,1]$.

3. Space of fuzzy random variables

For all $X, Y \in \Lambda$ we can define the function $\mathcal{D} : \Lambda \times \Lambda \to R$

$$\mathcal{D}(X,Y) = \sup_{\alpha \ge 0} \Delta(X_\alpha, Y_\alpha).$$

Two fuzzy variables $X, Y \in \Lambda$ are considered to be identical if $\mathcal{D}(X,Y) = 0$. It is obvious that $\mathcal{D}$ is a metric in Λ since Δ is metric in $\mathcal{L}$ (Th. 3.3 [4]).

Theorem 1. *$(\Lambda, \mathcal{D})$ is a complete metric space.*

Proof. Since Δ is metric in $\mathcal{L}$ we have that $\mathcal{D}$ is metric too. Let $\{X^n\}_{n\in N}$ be a Cauchy sequence in Λ, that is, for every $\varepsilon > 0$ there exists $n_0(\varepsilon)$

such that $\mathcal{D}(X^n, X^k) < \varepsilon$ for all $n, k > n_0(\varepsilon)$. For every $\alpha \in [0,1]$ the sequence $\{X^n_\alpha\}_{n\in N}$ is a Cauchy sequence in complete metric space $(\mathcal{L}, \Delta)$ and $\lim_{n\to\infty} X^n_\alpha = \tilde{X}_\alpha \in \mathcal{L}$. Let $S^n_\alpha = S_{X^n_\alpha}$ and $S_\alpha = S_{\tilde{X}_\alpha}$. We can define the random set X_α by

$$X_\alpha(\omega) = cl\{f(\omega) \; : \; f \in S_\alpha\}.$$

It is obvious that $\Delta(X_\alpha, \tilde{X}_\alpha) = 0$ which implies that $lim_{n\to\infty} X_\alpha \in \mathcal{L}$. Let $X(\omega)(x) = \sup_{\alpha>0} \{x \in X_\alpha(\omega)\}$. We shall show that for every $\omega \in \Omega$ all the conditions of Lemma 1 are satisfied which will mean that X is a fuzzy random variable.

1. By $X^n_\theta(\omega)$ we denote the set $\{x \in R^n : X^n(\omega)(x) \geq 0\}$. Since $X^n_\theta(\omega) = R^n$ for all $n \in N$ and all $\omega \in \Omega$ we have that $\lim_{n\to\infty} X^n_\theta(\omega) = R^n$ for all $\omega \in \Omega$.

2. For every $\omega \in \Omega$ and every $n \in N, X^n(\omega)$ is a fuzzy set. The inequality $\alpha < \beta$ implies $X^n_\beta(\omega) \subseteq X^n_\alpha(\omega)$, that is, $S^n_\beta \subseteq S^n_\alpha$. Since $S_\beta = \lim_{n\to\infty} S^n_\beta \subseteq \lim_{n\to\infty} S^n_\alpha = S_\alpha$ we have

$$X_\alpha(\omega) = cl\{f(\omega) : f \in S_\alpha\} \subseteq \{f(\omega) : f \in S_\beta\} = X_\beta(\omega),$$

for all $\omega \in \Omega$.

3. Let $\{\alpha_i\}_{i\in N} \subset (0,1]$ be a nondecreasing sequence and let $\lim_{i\to\infty} \alpha_i = \alpha$. In order to prove that for every $\omega \in \Omega$ $X_\alpha(\omega) = \bigcap_{i=1}^\infty X_{\alpha_i}(\omega)$ we shall prove that $S_\alpha = \bigcap_{i=1}^\infty S_{\alpha_i}$. As it was already proved, from $\alpha_i \leq \alpha$ we get $S_\alpha \subseteq S_{\alpha_i}, S_\alpha \subseteq \bigcap_{i=1}^\infty S_{\alpha_i}$. Using the Hausdorff semimetric r $(r(A,B) = \sup_{a\in A} \inf_{b\in B} \| a - b \|, \; A, B \subseteq L)$ we get

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S_\alpha) \leq r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, \bigcap_{i=1}^{\infty} S^j_{\alpha_i}) + r(\bigcap_{i=1}^{\infty} S^j_{\alpha_i}, S^j_\alpha) + r(S^j_\alpha, S_\alpha)$$

for a fixed j. But, $r(\bigcap_{i=1}^\infty S^j_{\alpha_i}, S^j_\alpha) = 0$, consequently, for every $\varepsilon > 0$, there exists j_0 such that

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S) \leq \varepsilon + r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S^j_{\alpha_i})$$

for $j > j_0$, since $S^j_\alpha \to S_\alpha$ uniformly in $\alpha \in [0,1]$.
Now

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, \bigcap_{i=1}^{\infty} S^j_{\alpha_i}) \leq r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S_{\alpha_n}) + r(S_{\alpha_n}, S^j_{\alpha_n}) + + r(S^j_{\alpha_n}, \bigcap_{i=1}^{\infty} S^j_{\alpha_i})$$

for any $n \geq 1$. Since $\bigcap_{i=1}^{\infty} S_{\alpha_i} \subseteq S_{\alpha_n}$, we obtain

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, \bigcap_{i=1}^{\infty} S^j_{\alpha_i}) \leq r(S_{\alpha_n}, S^j_{\alpha_n}) + r(S^j_{\alpha_n}, \bigcap_{i=1}^{\infty} S^j_{\alpha_i}).$$

Now $r(S_{\alpha_n}, S^j_{\alpha_n}) < \varepsilon$ for $j > j_0$. Note that (since the convergence $S^j_\alpha \rightarrow S_\alpha$ is uniform in α) j_0 does not depend on n. Since $S^j_{\alpha_n}$ decreases to $\bigcap_{i=1}^{\infty} S^j_{\alpha_i}$ when $n \rightarrow \infty$, it follows that

$$r(S^j_{\alpha_{n_0}}, \bigcap_{i=1}^{\infty} S^j_{\alpha_n}) < \varepsilon \text{ for some } n_0 \text{ (depending on } j).$$

Thus

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S^j_{\alpha_i}) < 2\varepsilon \text{ for } j > j_0, \text{ that is,}$$

$$r(\bigcap_{i=1}^{\infty} S_{\alpha_i}, S_\alpha) \leq 3\varepsilon \Rightarrow \bigcap_{i=1}^{\infty} S_{\alpha_i} \subseteq S_\alpha.$$

So we have proved that $S_\alpha = \bigcap_{i=1}^{\infty} S_{\alpha_i}$.

Now we have to prove that $X_\alpha(\omega) = \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega)$ for all $\omega \in \Omega$.We know that $X_\beta(\omega)$ is compact for all $\beta \in [0,1]$ and $\omega \in \Omega$. Let $Z(\omega) = \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega)$, $\omega \in \Omega$. Since the family of sets $X_{\alpha_i}(\omega)$ has the finite intersection property (every finite subsystem has a non-void intersection), $Z(\omega) \neq 0$ and $Z(\omega)$ is a compact set. Further, we have the next implication

$$\alpha \geq \alpha_i \Rightarrow X_\alpha(\omega) \subseteq X_{\alpha_i}(\omega) \Rightarrow X_\alpha(\omega) \subseteq \bigcap_{i=1}^{\infty} X_{\alpha_i}(\omega),$$

that is

(1) $$X_\alpha(\omega) \subseteq Z(\omega) \text{ for all } \omega \in \Omega.$$

Since $Z \in \mathcal{L}$ it follows (*Lemma* 1.1 [4]) that there exists the sequence $\{f_n\} \subset L$ such that $Z(\omega) = \text{cl } \{f_n(\omega)\}$ for all $\omega \in \Omega$,and we get

$$Z(\omega) \subseteq X_{\alpha_i}(\omega) \Rightarrow \{f_n\} \subseteq S_{\alpha_i} \Rightarrow \{f_n\} \subseteq \bigcap_{i=1}^{\infty} S_{\alpha_i} = S_\alpha \Rightarrow$$

$$\text{cl } \{f_n(\omega)\} \subseteq \text{cl } \{g(\omega) : g \in S_\alpha\} \text{ if } \omega \in \Omega \setminus A$$

and

$$\text{cl } \{f_n(\omega)\} = B \text{ if } \omega \in A,$$

which means that

(2) $$Z(\omega) \subseteq X_\alpha(\omega).$$

Lemma 1 is applicable and there exists fuzzy random variable X with $[X(\omega)]_\alpha = X_\alpha(\omega)$ for every $\alpha \in [0,1]$. From the completness of $(\mathcal{L}, \Delta)$ it follows that $X_\alpha \in \mathcal{L}$ for all $\alpha \in (0,1]$, which shows that $X \in \Lambda$.

It remains to show that $X^n \to X$ in $(\Lambda, \mathcal{D})$. For $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that $\mathcal{D}(X^n, X^k) < \varepsilon,\ \ n,k > n_0(\varepsilon)$. Let $n > n_0$ be fixed. Then

$$\Delta(X^n_\alpha, X_\alpha) = \lim_{k\to\infty} \Delta(X^n_\alpha, X^k_\alpha) \le$$

$$\lim_{k\to\infty} \sup_{\alpha>0} \Delta(X^n_\alpha, X^k_\alpha) = \lim_{k\to\infty} \mathcal{D}(X^n, X^k) < \varepsilon.$$

Hence $\sup_{\alpha>0} \Delta(X^n_\alpha, X_\alpha) \le \varepsilon$ for all $n > n_0(\varepsilon)$, implying that $X^n \to X$ in the metric $\mathcal{D}$. That completes the proof.

4. Conditional expectation

Motivated by definition of conditional expectation for random set we introduce the notion of fuzzy conditional expectation for fuzzy random variable.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{F}$ a sub–σ–algebra of $\mathcal{A}$ and $F \in \mathcal{L}$. The conditional expectation of F with respect to $\mathcal{F}$, which is in $\mathcal{L}(\Omega, \mathcal{F})$, is determined by setting

$$S_{E(F|\mathcal{F})} = cl\{g \in L(\Omega, \mathcal{F}) : g = E(f|\mathcal{F}),\quad f \in S_F\}.$$

Finally if X is a fuzzy random set we can define the conditional expectation of $X \in \Lambda$ in such a way that the following conditions are satisfied:

$$E(X|\mathcal{F}) \in \Lambda(\Omega, \mathcal{F}),$$

$$\{x \in R^n : E(X|\mathcal{F})(\omega)(x) \ge \alpha\} = E(X_\alpha(\omega)\,|\mathcal{F}).$$

The next theorem shows that there exists a unique fuzzy random variable satisfying these requirements. The proof is based on Lemma 1.

Theorem 2. *If $X \in \Lambda(\Omega, \mathcal{A})$, then there exists a unique fuzzy random variable $Y \in \Lambda(\Omega, \mathcal{F})$ such that*

$$Y_\alpha(\omega) = E(X_\alpha(\omega)\ |\mathcal{F}).$$

Proof.

1. Since $X_\theta(\omega) = R^n$ for every $\omega \in \Omega$ then $S_{X_\theta} = L(\Omega, \mathcal{A})$ and $S_{E(X_\theta|\mathcal{F})} = \tilde{S}_\theta = L(\Omega, \mathcal{F})$, which means that $\{f(\omega) : f \in L(\Omega, \mathcal{F})\} = R^n = Y(\omega)$ for all $\omega \in \Omega$.

2. If $\alpha \leq \beta$, then, clearly, $S_{X_\alpha} = S_\alpha \supseteq S_\beta = S_{X_\beta}$ which implies that $\tilde{S}_\alpha = S_{E(X_\alpha|\mathcal{F})} \supseteq S_{E(X_\beta|\mathcal{F})} = \tilde{S}_\beta$, and $Y_\alpha(\omega) \supseteq Y_\beta(\omega)$ for all $\omega \in \Omega$.

3. Now, we suppose that $\alpha_1 \leq \alpha_2 \ldots, \lim_{n\to\infty} \alpha_n = \alpha$. We denote $S_{X_{\alpha_i}}, S_{X_\alpha}, S_{E(X_{\alpha_i}|\mathcal{F})}$ and $S_{E(X_\alpha|\mathcal{F})}$ by $S_i, S, \tilde{S}_i$ and $\tilde{S}$ respectively. Our aim is to show that $\tilde{S} = \bigcap_{i=1}^\infty \tilde{S}_i$. Since $\tilde{S} \subseteq S_i$ it follows $\tilde{S} \subseteq \bigcap_{i=1}^\infty \tilde{S}_i$. But, from the nonexpansivity of conditional expectation for every $\varepsilon > 0$ there exists $i_0(\varepsilon)$ such that $h(\tilde{S}_i, \tilde{S}) \leq h(S_i, S) < \varepsilon$ for all $i > i_0(\varepsilon)$.

In order to prove that $Y_\alpha(\omega) = \bigcap_{i=1}^\infty Y_{\alpha_i}(\omega)$ for all $\omega \in \Omega$ we proceed as follows. If $\omega \in \Omega$, then $Y_\beta(\omega) \subseteq Y_0(\omega) \in K_{cc}(\mathcal{X})$ wich implies that $Y_\beta(\omega) \neq \emptyset$ is compact for all $\beta \in [0,1]$. Let

$$Z(\omega) = \bigcap_{i=1}^{\infty} Y_{\alpha_i}(\omega) \text{ for all } \omega \in \Omega$$

(that is $H(Y_{\alpha_i}(\omega), Z(\omega)) \to 0$ for all $\omega \in \Omega$), and let $\{f_n\} \in S_Z(\mathcal{F})$ be the sequence (*Lemma* 1.1 [4]) such that

$$Z(\omega) = \text{ cl } \{f_n(\omega)\} \text{ for all } \omega \in \Omega.$$

Since $Z(\omega) \subseteq Y_{\alpha_i}(\omega)$ for all $\omega \in \Omega$, we have that

$$\{f_n\} \subseteq \tilde{S}_{\alpha_i} \text{ for all } i \in N,$$

that is

$$\{f_n\} \subseteq \bigcap_{i=1}^{\infty} \tilde{S}_i = \tilde{S}.$$

Then

$$Z(\omega) = \text{ cl } \{f_n(\omega)\} \subseteq \text{ cl } \{g(\omega) : g \in \tilde{S}\} = Y_\alpha(\omega).$$

On the other hand, from $\alpha \geq \alpha_i$ for all $i \in N$, we get

$$Y_\alpha(\omega) \subseteq Y_{\alpha_i}(\omega) \Rightarrow Y_\alpha(\omega) \subseteq \bigcap_{i=1}^{\infty} Y_{\alpha_i}(\omega) = Z(\omega).$$

From last two relations we get that

$$Y_\alpha(\omega) = \bigcap_{i=1}^{\infty} Y_{\alpha_i}(\omega) \text{ for all } \omega \in \Omega.$$

Since $Y_\alpha = \mathcal{E}[X_\alpha|\mathcal{F}]$ we have (Th. 5.6. [4]) that $Y_\alpha \in \mathcal{L}(\Omega, \mathcal{F},)$ which implies that Y is integrably bounded. The uniqueness of Y follows from the uniqueness of conditional expectation of random sets.

The fuzzy random variable $Y \in \Lambda(\Omega, \mathcal{F},)$ defined below is fuzzy conditional expectation of $X \in \Lambda(\Omega, \mathcal{A},)$. We shall use notation $Y = E(X|\mathcal{F})$.

Theorem 3. *The fuzzy conditional expectation has the following properties:*

1. $\mathcal{D}(E(X_1|\mathcal{F}), E(X_2|\mathcal{F})) \le \mathcal{D}(X_1, X_2)$ *for all* $X_1, X_2 \in \Lambda$.

2. If $\mathcal{F}_1 \subset \mathcal{F} \subset \mathcal{A}$ *and* $X \in \Lambda$, *then* $E(X|\mathcal{F}_1)$ *taken on the base space* $(\Omega, \mathcal{A}, P)$ *is equal to* $E(X|\mathcal{F}_1)$ *taken on the base space* $(\Omega, \mathcal{F}, P)$.

3. If $\mathcal{F}_1 \subset \mathcal{F} \subset \mathcal{A}$ *and* $X \in \Lambda$, *then* $E(E(X|\mathcal{F})|\mathcal{F}_1) = E(X|\mathcal{F}_1)$.

4. If $X_n : \Omega \to \mathcal{F}(R^n)$ *are uniformly integrable bounded and* $X_n \xrightarrow{\mathcal{D}} X$, *then* $E(X_n|\mathcal{F}) \xrightarrow{\mathcal{D}} E(X|\mathcal{F})$.

This theorem is a fuzzy generalization of Th. 5.2. (1), Th. 5.3. [4] and Th. 6.2. [8]. The proof is quite similar to the case of random sets so it is omited.

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REZIME

RASPLINUTA SLUČAJNA PROMENLJIVA SA VREDNOSTIMA U R^n

Nad skupom integrabilno ograničenih fazi slučajno promenljivih je definisana metrika i dokazano je da je prostor kompletan. Dalje se definiše i ispituju se osobine uslovnog matematičkog očekivanja fazi slučajne promenljive.

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