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ON POWERS OF HEAVISIDE'S FUNCTION

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Abstract

We show that the non-associativity of the distribution product must be taken into account when differentiating powers of Heaviside's function.

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Heaviside's function is the locally summable function Y defined by

$$Y(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

It follows that

$$Y^n = Y. \tag{1}$$

Formal differentiation of equation (1) appears to give

$$nY^{n-1}\delta = \delta = nY\delta \tag{2}$$

and so

$$Y\delta = \frac{1}{n}\delta$$

for $n = 2, 3, \ldots$, which is clearly impossible.

However, it is well known that the product of distributions is in general non-associative and taking this into account, equation (2) is false, since $(Y^n)' \neq nY^{n-1}\delta$ for $n = 3, 4, \ldots$. It is desirable and indeed possible for equation (1) to hold for $n = 2, 3, \ldots$ and for formal differentiation of the product Y^n to give δ. Indeed, if ϕ is an arbitrary test function with compact support, then

$$\langle Y^n, \phi \rangle = \langle Y, \phi \rangle$$

for $n = 2, 3, \ldots$ and so equation (1) follows.

A definition for a commutative product of distributions was given in [1] and a definition for a non-commutative product of distributions was given in [2]. Both definitions lead to the result that

$$Y\delta = \delta Y = \frac{1}{2}\delta \tag{3}$$

and so we will accept equation (3).

It is easily seen that both of these definitions lead to a non-associative product since with either definition we have

$$Y(Y\delta) = \frac{1}{2}Y\delta = \frac{1}{4}\delta$$

whereas

$$(YY)\delta = Y\delta = \frac{1}{2}\delta.$$

The non-associativity of the product must therefore be taken into account when differentiating Y^n.

We will now consider a product of n distributions $f_1, \ldots, f_n$. We will further suppose that the product exists and is taken in the form

$$f_1(f_2 \cdots (f_{n-1}f_n) \cdots). \tag{4}$$

If the resulting products exist, formal differentiation of (4) gives

$$\begin{aligned}[f_1(f_2 \cdots (f_{n-1}f_n) \cdots)]' \;=\; & f_1'(f_2 \cdots (f_{n-1}f_n) \cdots) + f_1(f_2' \cdots \\ & \cdots (f_{n-1}f_n) \cdots) + \cdots + f_1(f_2 \cdots f_{n-1}' f_n) \cdots) + \\ & + f_1(f_2 \cdots (f_{n-1}f_n') \cdots).\end{aligned}$$

In particular, when

$$f_1 = \ldots = f_n = Y,$$

we have

$$\begin{aligned}[][Y(Y\ldots(YY)\ldots)]' &= \delta(Y\ldots(YY)\ldots)+Y(\delta\ldots(YY)\ldots)+\ldots+\\ &\quad +Y(Y\ldots(\delta Y)\ldots)+Y(Y\ldots(Y\delta)\ldots)\\ &= \delta Y+Y(\delta Y)+Y(Y(\delta Y))+\ldots+Y(Y\ldots\\ &\quad \ldots(\delta Y)\ldots)+Y(Y\ldots(Y\delta)\ldots)\\ &= \frac{1}{2}\delta+(\frac{1}{2})^2\delta+(\frac{1}{2})^3\delta+\ldots+(\frac{1}{2})^{n-1}\delta+(\frac{1}{2})^{n-1}\delta\\ &= \delta = Y'.\end{aligned}$$

We therefore see that it is equation (2) that is false.

Notice that in the above, the product Y^n has been taken in the form

$$Y^n = Y(Y\ldots(YY)\ldots) \tag{5}$$

to obtain the result

$$(Y^n)' = \delta. \tag{6}$$

However, the product Y^n could have been formed in many different ways. For example, we could have

$$Y^n = Y^r Y^{n-r},$$

where $1 < r < n$ and Y^r and Y^{n-r} are of the form (5), but whichever way the product Y^n is formed, equation (6) will always hold.

To see this, assume that for some $n > 1$

$$(Y^r)' = \delta$$

for $r = 2,3,\ldots,n$ for whichever way the product Y^r is formed. This is certainly true when $n = 2$. The product Y^{n+1} must be of one of the following three forms

$$Y(Y^n), \quad (Y^n)Y, \quad (Y^r)(YY^{n-r}),$$

where $1 < r < n$. By our assumption,

$$(Y^n)' = (Y^r)' = (Y^{n-r})' = \delta,$$

whichever way the products Y^n, Y^r, Y^{n-r} have been formed. Thus

$$(Y(Y^n))' = \delta(Y^n) + Y(Y^n)' = \delta Y + Y\delta = \delta,$$

$$((Y^n)Y)' = (Y^n)'Y + (Y^n)\delta = \delta Y + Y\delta = \delta$$

and

$$\begin{aligned}[(Y^r)(Y(Y^{n-r}))]' &= (Y^r)'(Y(Y^{n-r})) + (Y^r)(\delta(Y^{n-r})) + (Y^r)(Y(Y^{n-r})') \\ &= \delta Y + Y(\delta Y) + Y(Y\delta) \\ &= \frac{1}{2}\delta + (\frac{1}{2})^2\delta + (\frac{1}{2})^2\delta = \delta.\end{aligned}$$

It follows that however the product Y^{n+1} has been formed

$$(Y^{n+1})' = \delta$$

and our result follows by induction.

References

[1] Fisher, B.: The product of distributions, Quart. J. Math. (2), 22 (1971), 291-298.

[2] Fisher, B.: On defining the product of distributions, Math. Nachr., 99 (1980), 239-249.

REZIME

O STEPENIMA HEAVISIDEOVE FUNKCIJE

Autor pokazuje da se neasocijativnost proizvoda distribucija mora uzeti u obzir kod diferenciranja stepena Heavisideove funkcije.

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