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REGULAR BOREL t-DECOMPOSABLE MEASURES

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Abstract

The aim of the present paper is to study the regularity of Borel t-conorm decomposable measures.

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1 . Introduction

In this paper we shall investigate the regularity property of the $\perp$-decomposable measure with respect to a t-conorm $\perp$. We have considered such measures in previous papers [6]- [8] (see also [1] and [10]) on families of abstract sets. In this paper we shall restrict ourselves to the $\perp$-decomposable measures defined on the class $\mathcal{B}$ of Borel sets of a locally compact set X. Among other results, we shall prove the singleton characterization of atoms of the regular Borel $\perp$-decomposable measure - Theorem 1. We shall also give some conditions which ensure the regularity of the $\perp$-decomposable measures with respect to continuous at zero t-conorms $\perp$-Theorem 4. We shall prove the order continuity of $\perp$-decomposable measures which are decomposable with respect to t-conorm $\perp$, which have a series property - Theorem 5.

2 . Consequences of the regularity

Let X be locally compact Hausdorff topological space and let $\mathcal{K}$ be the lattice of all the compact subsets of X. Borel σ-ring $\mathcal{B}$ is the smallest σ-ring containing $\mathcal{K}$. We shall denote by $\mathcal{O}$ the class of all the open sets belonging to $\mathcal{B}$.

A function $\perp : [0,1] \times [0,1] \to [0,1]$ will be called t-conorm, if it is asociative, commutative, monotone and has 0 as a neutral element (see [1], [6] - [8], [10]). A set function $m : \mathcal{B} \to [0,1]$ with $m(\emptyset) = 0$ will be caled a (Borel) $\perp$-decomposable measure if

$$m(A \cup B) = m(A) \perp m(B)$$

holds for all $A, B \in \mathcal{B}$, such that $A \cap B = \emptyset$.

A $\perp$-decomposable measure m is **regular**, if for each set $A \in \mathcal{B}$ and each $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $V \in \mathcal{O}$, such that $K \subset A \subset V$ and

$$m(V \setminus K) < \varepsilon.$$

A set function $m : \mathcal{B} \to [0,1]$ is **order continuous** if

$$\lim_{n\to\infty} m(E_n) = 0$$

for any sequence (E_n), $E_n \in \mathcal{B}$ $(n \in \mathcal{N})$, such that $E_n \searrow \emptyset$.

A set function $m : \mathcal{B} \to [0,1]$ is **exhaustive**, if

$$\lim_{n\to\infty} m(E_n) = 0$$

for any sequence (E_n) of pairwise disjoint sets from $\mathcal{B}$.

A set $A \in \mathcal{B}$ is an **atom** of $\perp$-decomposable measure m iff $m(A) > 0$ and either $m(A \cap B) = 0$ or $m(A \setminus B) = 0$ for any $B \in \mathcal{B}$.

We have in the following theorem as an important property of atoms of regular Borel $\perp$-decomposable measures.

Theorem 1. *Let m: $\mathcal{B} \to [0,1]$ be a regular Borel $\perp$-decomposable measure with respect to an arbitrary but fixed t-conorm $\perp$. If $A \in \mathcal{B}$ is an atom of m, then there exists a point $a \in A$ such that*

$$m(A) = m(\{a\}).$$

Proof. Let $A \in \mathcal{B}$ be an arbitrary, but fixed atom of m. We denote with $\mathcal{K}'$ the family of the compact sets $K \subset A$, such that $m(A \setminus K) = 0$. Since for of any fixed $K \in \mathcal{K}'$,

$$K \cap B \subset A \cap B \text{ and } K \setminus B \subset A \setminus B$$

hold for any $B \in \mathcal{B}$, we have by the monotonicity of m

$$m(K \cap B) \cdot m(K \setminus B) = 0.$$

Then, since we have

$$m(K) = m(A \setminus K) \perp m(K) = m(A) > 0 \,,$$

K is an atom of m.

Further, for each K_1 and K_2 from $\mathcal{K}'$ we have by the monotonicity and $\perp$-subaddivity of m

$$m(A \setminus (K_1 \cap K_2)) = m((A \setminus K_1) \cup (A \setminus K_2)) \leq$$

$$\leq \; m(A \setminus K_1) \perp m(A \setminus K_2).$$

Hence, by $m(A \setminus K_i) = 0 \;\; (i = 1, 2)$ we have

$$m(A \setminus (K_1 \cap K_2)) = 0, \text{ i.e. } K_1 \cap K_2 \in \mathcal{K}'.$$

Now, let

$$K_0 = \bigcap_{K \in \mathcal{K}'} K.$$

Then K_0 is a non-emty compact set. Indeed, if we would suppose the contrary, i.e. $K_0 = \emptyset$, then, since X is a Hausdorff topological space, there would exist some finite subcollection of $\{K\}_{K \in \mathcal{K}'}$ with an empty intersection. This is imposible, since this finite subcollection would belong to $\mathcal{K}'$, but it is an atom as an element of $\mathcal{K}'$, which is non-empty. We shall show that $K_0 \in \mathcal{K}'$. Namely, for an arbitrary but fixed $K \in \mathcal{K}'$ there has to be $m(K \setminus K_0) = 0$. If we suppose that this is not true, then, since for any $B \in \mathcal{B}$, we have

$$B \cap (K \setminus K_0) \subset A \cap B \text{ and}$$

$$(K \setminus K_0) \setminus B \subset A \setminus B,$$

we obtain

$$m(B \cap (K \setminus K_0)) \cdot m((K \setminus K_0) \setminus B) = 0,$$

i.e. $K \setminus K_0$ would be an atom of m. We have

$$m(A) = m(A \setminus K) \perp m(K \setminus K_0) \perp m(K_0). \tag{1}$$

Since we supposed that $m(K \setminus K_0) > 0$ and A , K are atoms of m (which implies $m(A) > 0$ and $m(K_0) = m(K \cap K_0) = 0$) , we obtain by (1)

$$m(A) = m(K \setminus K_0).$$

Hence,

$$m(A \setminus (K \setminus K_0)) = 0.$$

These facts imply that $K \setminus K_0$ has to contain an element of $\mathcal{K}'$. Since K_0 is non-empty, this is a contradiction. So, we have $m(K \setminus K_0) = 0$. Then, by (1), we obtain $m(A) = m(K_0)$, i.e. $m(A \setminus K_0) = 0$. That means $K_0 \in \mathcal{K}'$. We shall show that K_0 reduces on a set with one element. If we suppose the contrary that K_0 contains at least two distinct elements a_1 and a_2, then, since X is a locally compact Hausdorff topological space, there exists an open neighbourhood V of a_1 such that $\bar{V}$ does not contain a_2 . Then, we have

$$K_0 = (K_0 \setminus V) \cup (K_0 \cap \bar{V}).$$

Since one of the sets $K_0 \setminus V$, $K_0 \cap \bar{V}$ has to belong to $\mathcal{K}'$, but K_0 is the least element from $\mathcal{K}'$, we obtain a contradiction. So, we have for some $a \in A \quad m(A) = m(K_0) = m(\{a\})$.

In the special case $X = \mathbf{R}$, we have

Theorem 2. *Each continuous from above regular Borel $\perp$-decomposable measure m: $\mathcal{B}(\mathbf{R}) \to [0,1]$ with respect to an arbitrary but fixed t-conorm $\perp$ on the Borel σ-algebra $\mathcal{B}(\mathbf{R})$, which has the property*

$$m((a,b)) = g(b-a) \quad ((a,b) \subset [0,1])$$

for some continuous at 0 function g with $g(0) = 0$, is non-atomic.

Proof. Suppose the contrary. Then, there exists an atom $A \in \mathcal{B}(\mathbf{R})$ of m. Then, by Theorem 1, there exists $a \in A$, such that $m(A) = m(\{a\})$. Then,

$$m(\{a\}) = m(\bigcap_{n=1}^{\infty}(a - \frac{1}{n}, a + \frac{1}{n})) =$$

$$= \lim_{n\to\infty} m((a - \frac{1}{n}, a + \frac{1}{n})) =$$

$$= \lim_{n\to\infty} g(\frac{2}{n}) = 0 \text{ i.e. } m(\{a\}) = 0.$$

Contradicton.

Theorem 3. *A regular Borel $\perp$-decomposable measure m with respect to an arbitrary but fixed t-conorm $\perp$ is exhaustive on the family of open sets $\mathcal{O}$.*

Proof. Let (O_n) be a sequence of open sets from $\mathcal{O}$ which are pairwise disjoint. Since $\cup_{n=1}^{\infty} O_n$ is an open set, for any $\varepsilon > 0$, there exists a compact set K such that $K \subset \cup_{n=1}^{\infty} O_n$ and

$$m(\bigcup_{n=1}^{\infty} O_n \setminus K) < \varepsilon.$$

Since (O_n) is an open cover of K, there exists a natural number n_0 such that $K \subset \cup_{n=1}^{n_0} O_n$. Then, we have by the monotonicity of m for $k \geq n_0 + 1$

$$m(O_k) \leq m(O_k \bigcup (\bigcup_{n=1}^{n_0} O_n \setminus K)) \leq m(\bigcup_{n=1}^{\infty} O_n \setminus K)$$

3 . Regularity and continuity

We have in the following theorem conditions which ensure the regularity of the Borel t-conorm decomposable measure.

Theorem 4. *Let m: $\mathcal{B} \to [0,1]$ be a $\perp$-decomposable measure with respect to a continuous at zero t-conorm $\perp$, which is exhaustive on the family of compact sets $\mathcal{K}$ and satisfies the condition*

$$m(A) = sup\{m(K) : K \in \mathcal{K}, K \subset A\} \quad (A \in \mathcal{B}). \tag{2}$$

Then m is a regular Borel $\perp$-decomposable measure.

Proof. Suppose that m is not regular. Then, there exist a set A from $\mathcal{B}$ and a number $\varepsilon > 0$ such that

$$m(V \setminus K) > \varepsilon \tag{3}$$

for each $K \in \mathcal{K}, V \in \mathcal{O}, K \subset A \subset V$. Let us fix such sets K_0 and V_0. Then, by (2), there exist $C_1, D_1 \in \mathcal{K}$, $C \subset V_0 \setminus A$, $D_1 \subset A \setminus K_0$, such that

$$m(C_1) \geq \frac{1}{2} m(V_0 \setminus A) \text{ and } m(D_1) \geq \frac{1}{2} m(A \setminus K_0).$$

If we denote by $V_1 = V_0 \setminus C_1$ and $K_1 = K_0 \cup D_1$, then we have $V_1 \in \mathcal{O}$ and $K_1 \in \mathcal{K}$ and

$$K_1 \subset A \subset V_1.$$

Hence, by (3), we have

$$m(V_1 \setminus A_1) > \varepsilon.$$

Now, we are going to repeat the preceding procedure. After n-steps we have

$$C_n, D_n \in \mathcal{K},\ C_n \subset V_{n-1} \setminus A,\ D_n \subset A \setminus K_{n-1},$$

such that

$$m(C_n) \geq \frac{1}{2} m(V_{n-1} \setminus A); \quad m(D_n) \geq \frac{1}{2} m(A \setminus K_{n-1}). \tag{4}$$

So, we have two sequences (C_n) and (D_n) of pairwise disjoint sets from $\mathcal{K}$, whice are subsets of V_0 . Since m is exhaustive on $\mathcal{K}$,we obtain by (4)

$$\lim_{n\to\infty} m(V_n \setminus A) = 0 \text{ and } \lim_{n\to\infty} m(A \setminus K_{n-1}) = 0.$$

Hence, by continuity at zero of t-conorm $\perp$, we have

$$\lim_{n\to\infty} m(V_n \setminus K_n) = \lim_{n\to\infty}((V_n \setminus A) \cup (A \setminus K_n)) = 0.$$

Contradiction with (3).

Now, we are going to introduce an important condition for the t-conorm $\perp$.

A t-conorm $\perp$ has the **series property** if for some $k_0 \in N$

$$\perp(\frac{x}{2}, \frac{x}{2^{k_0}}) \leq x \quad (x \in [0,1])$$

holds. It is obvious that t-conorm with the series property is also continuous at zero.

Many important known t-conorms have the series property. For example: $\perp_m(x,y) = min(x+y,1)$, $U_\lambda(x,y) = min(x+y+\lambda xy, 1)$ for $\lambda > -1, S_p(x,y) = (x^p + y^p - x^p y^p)^{\frac{1}{p}}$, for fixed $p > 0$, etc.

Theorem 5. *Each regular Borel $\perp$-decomposable measure m with respect to a t-conorm $\perp$ which has the series property is order continuous and exhaustive.*

Proof. Let (A_n) be a sequence from $\mathcal{B}$ such that $A_n \searrow \emptyset$ and let $\varepsilon > 0$. Since m is regular, we can choose $K_k \in \mathcal{K}$ $(k \in N)$ such that

$$m(A_k \setminus K_k) < \frac{\varepsilon}{2^{(k-1)+k_0}} \quad (k \in N).$$

Hence, by the series property of $\perp$ and $\perp$-subadditivity of m, we obtain

$$m(\bigcup_{k=1}^{n}(A_k \setminus K_k)) \leq$$

$$\leq (\ldots((m(A_n \setminus K_n) \perp m(A_{n-1} \setminus K_{n-1})) \perp m(A_{n-2} \setminus K_{n-2})) \perp \ldots \perp m(A_1 \setminus K_1)) \leq$$

$$\leq (\ldots((\frac{\varepsilon}{2^{(n-1)+k_0}} \perp \frac{\varepsilon}{2^{(n-2)+k_0}}) \perp \frac{\varepsilon}{2^{(n-3)+k_0}}) \perp \ldots \perp \frac{\varepsilon}{2^{k_0}}) \leq$$

$$\leq (\ldots(\frac{\varepsilon}{2^{(n-2)}} \perp \frac{\varepsilon}{2^{(n-3)+k_0}}) \perp \ldots \perp \frac{\varepsilon}{2^{k_0}}) \leq \varepsilon.$$

We have $\cap_{i=1}^{n} K_i \in \mathcal{K}$ and $\cap_{i=1}^{n} K_i \subset A_n$. Since $A_n \searrow \emptyset$, we obtain $\cap_{i=1}^{n} K_i \searrow \emptyset$. Then, there exists a natural number n_0 such that $\cap_{i=1}^{n} K_i = \emptyset$ for $n \geq n_0$. Hence, for $n \geq n_0$

$$A_n = A_n \setminus \bigcap_{i=1}^{n} K_i = \bigcup_{i=1}^{n}(A_n \setminus K_i) \subset \bigcup_{i=1}^{n}(A_i \setminus K_i).$$

Then, by the monotonicity of m, we have

$$m(A_n) \leq m(\bigcup_{i=1}^{n}(A_i \setminus K_i)) < \varepsilon$$

for $n \geq n_0$, i.e. m is order continuous. If (E_n) is a sequence of pairwise disjoint sets from $\mathcal{B}$, then the inequality

$$m(E_n) \leq m(\bigcup_{k=n}^{\infty} E_k) \text{ holds}$$

Since $\bigcup_{k=n}^{\infty} E_k \searrow \emptyset$, the preceding inequality implies the exhaustivity of m.

Remark 1. *Theorem 5 implies that the set function $m(A) = sup_{x \in A} f(x)$, from Example 2.4 of [8], is not regular, although it is σ-sup-decomposable and the sup has the series property.*

Corollary 1. *Each regular Borel $\perp$-decomposable measure m with respect to an Archimedean t-conorm $\perp$, which has the series property, is $\sigma - \perp$-additive.*

Remark 2. *For $\perp$-decomposable measure with respect to continuous (specially Archimedean) t-conorm, we can apply the results of H. Weber [9] and P. Morales [5] on uniform semigroup valued measures.*

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REZIME

REGULARNE BORELOVE t-KONORMA DEKOMPOZABILNE MERE

U radu se ispituje regularnost $\perp$-dekompozabilnih mera u odnosu na t-konormu $\perp$ a koje su definisane na Borelovim skupovima lokalno kompaktnog Hausdorffovog topološkog prostora.

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