Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 20, 2 (1990), 121-130

Review of Research
Faculty of Science
Mathematics Series

EXTENSION OF THE CONTINUOUS t-CONORM DECOMPOSABLE MEASURE

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Abstract

The aim of the present paper is to study the extensions of the $\perp$-decomposable measure m, with respect to a continuous t-conorm $\perp$, from a ring $\mathcal{R}$ to the σ-ring Σ generated by $\mathcal{R}$ and to the class $\mathcal{R}_\sigma$ of sets which are limits of increasing sequences from $\mathcal{R}$.

AMS Mathematics Subject Classification (1980): 28A10, 28B10
Key words and phrases: t-conorm, $\perp$-decomposable measure, order continuous, exhaustive, $\perp$-subadditive set function

1. Introduction

This paper is a continuation of investigations on $\perp$-decomposable measures with respect to a t-conorm $\perp$, which were considered in the papers [5] - [8]. The interest for these non-additive set function (non-additive with respect to the usual addition of the real numbers) is growing. We have proved in previous papers [5] and [6] the analogies of classical measure theory theorems: Lebesgue decomposition, Saks decomposition, Darboux property, compactness of the range. In this paper we shall investigate the extension of the order continuous $\perp$-decomposable measure m, with respect to a continuous t-conorm $\perp$, from a ring $\mathcal{R}$ to the σ-ring Σ generated by $\mathcal{R}$ - Theorem 3.2.

The unique extension is monotone order continuous $\perp$-subadditive set function $\overline{m}$. It is interesting that without the supposition of the order continuity of m we lose the uniqueness of the extension - Example 3.3. We have proved in Theorem 3.1. that there exists a unique σ $\perp$-decomposable extension of the σ $\perp$-decomposable measure from $\mathcal{R}$ to the class $\mathcal{R}_\sigma$ of sets which are limits of increasing sequences from $\mathcal{R}$.

2. $\perp$-subadditive set functions

Definition 2.1. *A function* $\perp : [0,1] \times [0,1] \to [0,1]$ *will be called t-conorm if it satisfies:*

(A) $\perp(x,0) = \perp(0,x) = x \quad (x \in [0,1])$;

(B) if $x_1 \leq x_3$ and $x_2 \leq x_4$ for $x_1, x_2, x_3, x_4 \in [0,1]$ then $\perp(x_1,x_2) \leq \perp(x_3,x_4)$;

(C) $\perp(x,y) = \perp(y,x) \quad (x,y \in [0,1])$;

(D) $\perp(\perp(x,y),z) = \perp(x,\perp(y,z)) \quad (x,y,z \in [0,1])$.

A t-conorm $\perp$ will be called continuous at zero if it satisfies the condition

(E) for all sequences (x_n) and (y_n) such that $x_n, y_n \in [0,1]$

$$\text{and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0, \lim_{n\to\infty} \perp(x_n,y_n) = 0 \text{ holds.}$$

There are many important t-conorms: $\perp_m(x,y) = \min(x+y,1)$, $U_\lambda(x,y) = min(x+y+\lambda xy,\ 1)$ for $\lambda > -1$, $S_p(x,y) = (x^p + y^p - x^p y^p)^{1/p}$ for fixed $p > 0$, etc. (see [5], [8]).

There exist t-conorms which are continuous at zero but they are not continuous. For example,

$$\perp(x,y) = \begin{cases} max\{x,y\} & \text{for} \quad y \in [0,1/2) \\ max\{x,y\} & \text{for} \quad x \in [0,1/2) \\ 1 & \text{otherwise} \end{cases}$$

In the whole paper, $\mathcal{R}$ always denotes a ring of subsets of the given nonempty set X.

Definition 2.2. *A set function $m : \mathcal{R} \to [0,1]$ with $m(\emptyset) = 0$ will be called $\perp$-subadditive, if*

$$m(A \cup B) \leq m(A) \perp m(B)$$

holds for all $A, B \in \mathcal{R}$, such that $A \cap B = \emptyset$.
If in the preceding inequality equality holds, then m will be called $\perp$-decomposable measure.

A set function $\eta : \mathcal{R} \to [0,\infty]$ is called a submeasure if it is monotone nondecreasing, subadditive and $\eta(\emptyset) = 0$.

Theorem 2.3. *If m is a monotone $\perp$-subadditive set function on a ring $\mathcal{R}$ of sets, then the following hold*

(i) $m(A \cup B) \leq m(A) \perp m(B)$ *for arbitrary* $A, B \in \mathcal{R}$,

(ii) if $\perp$ *is continuous at zero, then* $m(A_n) + m(B_n) \to 0$ as $n \to \infty$, for $A_n, B_n \in \mathcal{R}$ $(n \in \mathbb{N})$ *implies*

$$m(A_n \cup B_n) \to 0$$

Proof.

(i) For $A \cap B = \emptyset$, the inequality is true by Definition 2.2. Let $A, B \in \mathcal{R}$ and $A \cap B \neq \emptyset$. Then, we have

$$m(A \cup B) = m(A \cup ((A \cup B)\backslash A)) \leq m(A) \perp m((A \cup B)\backslash A) \leq m(A) \perp m(B).$$

(ii) Follows by property (i).

A set function $m : \mathcal{R} \to [0,1]$ is order continuous (continuous from above at $\emptyset$), if $\lim_{n\to\infty} m(E_n) = 0$ for any sequence (E_n), $E_n \in \mathcal{R}\,(n \in \mathbb{N})$, such that $E_n \searrow \emptyset$

A set function $m : \mathcal{R} \to [0,1]$ is exhaustive, if $\lim_{n\to\infty}(E_n) = 0$ for any sequence (E_n) of pairwise disjoint sets from $\mathcal{R}$.

There exists $\sigma-$ $\perp$-decomposable measure with respect to a continuous t-conorm $\perp$, which is not order continuous.

Example 2.4. Let Σ be the Borel σ-algebra of subsets of the set of real numbers and let f be a continuous function on $\mathbb{R}$ such that $f(0) \neq 0$ and $0 \leq f(x) \leq 1 \ (n \in \mathbb{R})$. Then, the function

$$m(A) = \sup_{x \in A} f(x) \quad (A \in \Sigma)$$

is a $\sigma-\perp$-decomposable measure with respect to the continuous t-conorm $\perp = sup$, but it is not order continuous. Namely, if we take the sequence of open intervals $(0, \frac{1}{n})$, $n \in \mathbb{N}$, then

$$\bigcap_n (0, \frac{1}{n}) = \emptyset, \quad \text{but}$$

$$\lim_{n \to \infty} m((0, \frac{1}{n})) = \lim_{n \to \infty} \sup_{x \in (0, \frac{1}{n})} f(x) = f(0) \neq 0.$$

We have the following generalization of Theorem 3.2. from [5].

Theorem 2.5. *Let $m : \mathcal{R} \to [0,1]$ be a monotone $\perp$-subadditive set function with respect to a continuous at zero t-conorm $\perp$. Then, there exists a submeasure η on $\mathcal{R}$ such that*

$$\lim_{n \to \infty} m(E_n) = 0 \quad \text{iff} \lim_{n \to \infty} \eta(E_n) = 0.$$

The proof is quite analogous to the proof of Theorem 3.2. from [5], using Theorem 2.3. instead of Theorem 3.1. from [5].

Theorem 2.6. *A $\perp$-decomposable measure m on $\mathcal{R}$ is exhaustive iff for every monotone sequence (A_n) for $\mathcal{R}$ holds*

$$m(A_n \triangle A_m) \to 0 \quad \text{as} \quad n, m \to \infty,$$

where $A \triangle B = (A \backslash B) \cup (B \backslash A)$.

Proof. „ $\Rightarrow$" Suppose that the theorem is not true, i.e. that for nondecreasing sequence (A_n) there exist $\epsilon > 0$ and an increasing sequence (i_n) of natural numbers, such that

$$m(A_{i_{n+1}} \triangle A_{i_n}) > \epsilon \quad (n \in \mathbb{N}).$$

Since $(A_{i_{n+1}} \triangle A_{i_n})$ is a sequence of pairwise disjoint sets from $\mathcal{R}$, we have a contradiction with the exhaustivity of m.

„ $\Leftarrow$" Suppose now that m is not exhaustive. Then, there exists a sequence (E_n) of pairwise disjoint sets from $\mathcal{R}$ and $\epsilon > 0$ such that

$$m(E_n) > \epsilon \quad (n \in \mathbb{N}).$$

Let $A_n = \bigcup_{k=1}^{n} E_k$. Then, the sequence (A_n) is nondecreasing and since m is monotone, we have for $m > n$

$$m(A_n \Delta A_m) = m(\bigcup_{k=n}^{m} E_k) \geq m(E_n) > \epsilon.$$

Contradiction.

3. Extensions

Let $\mathcal{R}_\sigma$ be the class of sets which are limits of increasing sequences from $\mathcal{R}$.
Theorem 3.1. *Let $m : \mathcal{R} \to [0,1]$ be a $\sigma-\perp$-decomposable measure with respect to a continuous t-conorm $\perp$. Then, the function $m^+ : \mathcal{R}_\sigma \to [0,1]$ defined by*

$$m^+(A) := \lim_{n\to\infty} m(A_n) \qquad (A \in \mathcal{R}_\sigma), \tag{3.1}$$

where (A_n) is any sequence in $\mathcal{R}$ such that $A_n \nearrow A$, is a unique σ $\perp$-decomposable extension of m on $\mathcal{R}_\sigma$.

Proof. First we shall prove that m^+ is independent from the choice of the sequence (A_n) in (3.1.). Let $(A_n^{'})$ and $(A_n^{''})$ be two increasing sequences from $\mathcal{R}$ such that $A_n^{'} \nearrow A$ and $A^{''} \nearrow A$. Since m is continuous from below (Theorem 3.2 (iii) from [8]), we have

$$\lim_{n\to\infty}(\lim_{k\to\infty} m(A_n^{'} \cap A_k^{''})) = \lim_{n\to\infty} m(A_n^{'}) \tag{3.2}$$

and

$$\lim_{k\to\infty}(\lim_{n\to\infty} m(A_n^{'} \cap A_k^{''})) = \lim_{k\to\infty} m(A_k^{''}). \tag{3.3}$$

On the other hand, by the monotonicity of m we have

$$\lim_{n\to\infty}(\lim_{k\to\infty} m(A_n^{'} \cap A_k^{''})) \leq \lim_{k\to\infty} m(A_k^{''}) \tag{3.4}$$

and

$$\lim_{k\to\infty}(\lim_{n\to\infty} m(A_n^{'} \cap A_k^{''})) \leq \lim_{n\to\infty} m(A_n^{'}). \tag{3.5}$$

Then by (3.2) and (3.4) we have

$$\lim_{n\to\infty} m(A_n^{'}) \leq \lim_{k\to\infty} m(A_k^{''}), \tag{3.6}$$

and by (3.3) and (3.5) we obtain

(3.7) $\lim_{k\to\infty} m(A''_k) \le \lim_{n\to\infty} m(A'_n)$.

(3.6) and (3.7) imply

$$\lim_{n\to\infty} m(A'_n) = \lim_{n\to\infty} m(A''_n).$$

It is obvius that m^+ is an extension of m to $\mathcal{R}_\sigma$. We shall prove that m^+ is $\perp$-decomposable. Let $A, B \in \mathcal{R}_\sigma$, such that $A \cap B = \emptyset$. Let (A_n) and (B_n) be two sequences from such that $A_n \nearrow A$ and $B_n \nearrow B$. Hence, $A_n \cap B_n = \emptyset$ $(n \in \mathbb{N})$. Since the t-conorm $\perp$ is continuous, we have

$$m^+(A \cup B) =$$

$$= \lim_{n\to\infty} m(A_n \cup B_n) = \lim_{n\to\infty} (m(A_n) \perp m(B_n)) =$$

$$= \lim_{n\to\infty} m(A_n) \perp \lim_{n\to\infty} m(B_n) = m^+(A) \perp m^+(B).$$

m^+ is continuous from below, i.e. if $A, A_n \in \mathcal{R}_\sigma$ $(n \in \mathbb{N})$ and $A_n \nearrow A$, then $m^+(A) = \lim_{n\to\infty} m^+(A_n)$. Hence, by Theorem 3.2 (iii) from [8] follows that m^+ is $\sigma-$ $\perp$-decomposable.

Theorem 3.2. *Let $m : \mathcal{R} \to [0,1]$ be an order continuous $\perp$-decomposable measure with respect to a continuous t-conorm and let Σ be the σ-ring generated by $\mathcal{R}$. Then, m can be extended to a unique monotone order continuous $\perp$-subadditive set function $\overline{m} : \Sigma \to [0,1]$, iff the following conditions hold:*

(a) *If (A_n) is a sequence from $\mathcal{R}$ such that $m(A_n \triangle A_m) \to 0$ as $n, m \to \infty$, then there exists the limit of the sequence $(m(A_m))$,*

(b) *m is exhaustive.*

Proof First we shall prove that conditions (a) and (b) are sufficient. By Theorem 2.5. there exists an order continuous submeasure η or $\mathcal{R}$, such that $m(A_n) \to 0 \Leftrightarrow \eta(A_n) \to 0$. By 7.2. from [3] there exists an order continuous extension $\overline{\eta}$ of η to Σ.

We define the required extension $\overline{m}$ of m to Σ in the following way: for $A \in \Sigma$ we choose a sequence (A_n) from $\mathcal{R}$ such that $\overline{\eta}(A \triangle A_n) \to 0$ as $n \to \infty$ and we take

(3.8) $\overline{m}(A) = \lim_{n\to\infty} m(A_n)$.

The function $\overline{m}$ is independent of the choice of the sequence (A_n). It is obvious that $\overline{m}$ extends m. We shall prove that $\overline{m}$ is $\perp$-subadditive. Let $A, B \in \Sigma$, such that $A \cap B = \emptyset$. Let (A_n) and (B_n) be two sequences from $\mathcal{R}$, such that $\overline{\eta}(A \triangle A_n) \to 0$ and $\overline{\eta}(B \triangle B_n) \to 0$. Then we have $\overline{m}(A) = \lim_{n\to\infty} m(A_n)$ and $\overline{m}(B) = \lim_{n\to\infty} m(B_n)$.
Using the inclusion

$$(A \cup B)\triangle(A_n \cup B_n) \subset (A\triangle A_n) \cup (B\triangle B_n),$$

we have

$$\overline{\eta}((A \cup B)\triangle(A_n \cup B_n)) \leq \overline{\eta}((A\triangle A_n) \cup (B\triangle B_n)) \leq$$

$$\leq \overline{\eta}((A\triangle A_n)\setminus(B\triangle B_n)) + \overline{\eta}((B\triangle B_n) \leq$$

$$\leq \overline{\eta}((A\triangle A_n) + \overline{\eta}((B\triangle B_n),$$

where we have used that $\overline{\eta}$ is monotone and subadditive. The preceding inequality implies

$$\overline{\eta}((A \cup B)\triangle(A_n \cup B_n)) \to 0 \quad \text{as} \quad n \to \infty.$$

By the definition of the set function $\overline{m}$ (3.8), we have

$$\overline{m}(A \cup B) = \lim_{n\to\infty} m(A_n \cup B_n).$$

Hence, using the continuity of the t-conorm $\perp$

$$\overline{m}(A \cup B) = \lim_{n\to\infty} m(A_n \cup B_n) \leq \lim_{n\to\infty} (m(A_n) \perp m(B_n)) =$$

$$= \lim_{n\to\infty} m(A_n) \perp \lim_{n\to\infty} m(B_n) = \overline{m}(A) \perp \overline{m}(B).$$

The set function $\overline{m}$ is monotone. This follows by the fact that m^+ (for the definition see (3.1)) is monotone on $\mathcal{R}_\sigma$ and that for every $A \in \Sigma$ there exists a sequence (A_n) from $\mathcal{R}_\sigma$, such that $A \subset A_n \quad (n \in \mathbb{N})$, $A_n \searrow$ and

$$\overline{\eta}(A\triangle A_n) = \overline{\eta}(A_n\setminus A) \to 0.$$

Since $\overline{m}$ is monotone and $\perp$-subadditive, by Theorem 2.5., there exists an order continuous submeasure γ on Σ such that $\gamma(A_n) \to 0| \Leftrightarrow m(A_n) \to 0$.

Hence, the restriction of γ to $\mathcal{R}$, $\gamma|\mathbb{R}$, satisfies $(\gamma|\mathbb{R})(A_n) \to 0 \Leftrightarrow m(A_n) \to 0$. This implies by 7.3 from [3]

$$\gamma(A_n) \to 0 \Leftrightarrow \overline{\eta}(A_n) \to 0.$$

So, we have $\gamma \sim \overline{\eta}$. Since $\mathcal{R}$ is dense in the complete space $(\Sigma, \overline{\eta})$ (see [3], 3.1, 5.2 and 7.1), the set function $\overline{m}$ is by the condition (a) a continuous (unique) $\perp$-subadditive extension of m from $(\mathcal{R}, \eta)$ to $(\Sigma, \overline{\eta})$.

Since $\overline{m}$ is $\overline{\eta}$-continuous, $\overline{m}$ is order continuous.

Using the preceding facts about submeasures we can prove that condition (a) is also necessary.

If we suppose that $\overline{m}$ is the extension of m to Σ, then by Theorem 2.5 there exists an order continuous submeasure $\overline{\eta}$ on Σ such that

$$\overline{m}(E_n) \to 0 \Leftrightarrow \overline{\eta}(E_n) \to 0.$$

Since each order continuous submeasure on σ-ring is exhaustive, there follows by the preceding equivalence the exhaustivity of $\overline{m}$.

The assumption of the order continuity of the set function m in the preceding theorem is important. Namely, if we drop this assumption, then it can happen that the extension is not unique, as the following example shows.

Example 3.3. We are taking the sup-decomposable measure on the algebra $\mathcal{R}$ generated by closed intervals $[a, b]$, where a and b are rational numbers. We have for special functions f_1 and f_2, which are defined in the following way

$$f_1(x) = \begin{cases} \frac{1}{2} & \text{for} \quad x \in [0, i] \\ \frac{1}{3} & \text{for} \quad x \in (i, 1] \\ 0 & \text{otherwise} \end{cases}, \quad f_2(x) = \begin{cases} \frac{1}{2} & \text{for} \quad x \in [0, i) \\ \frac{1}{3} & \text{for} \quad x \in [i, 1] \\ 0 & \text{otherwise} \end{cases}$$

where i is some irational number from $(0, 1)$, the corresponding measures

$$m_1(A) = \sup_{x \in A} f_1(x) \ \text{ and } \ m_2(A) = \sup_{x \in A} f_2(x) \ \ (A \in \mathcal{R}).$$

Measures m_1 and m_2 are equal on $\mathcal{R}$. The algebra $\mathcal{R}$ generates the minimal σ-algebra Σ, Borel σ-algebra. m_1 is extended by $\overline{m_1}$ defined by

$$\overline{m_1}(A) = \sup_{x \in A} f_1(x) \quad (A \in \Sigma)$$

and m_2 is extended by $\overline{m_2}$ defined by

$$\overline{m_2}(A) = \sup_{x \in A} f_2(x) \quad (A \in \Sigma).$$

Although the sup-decomposable measures m_1 and m_2 are equal on the algebra $\mathcal{R}$, they have different extensions to the σ-algebra Σ. Namely, we have $m_1(\{i\}) = \frac{1}{2}$ and $m_2(\{i\}) = \frac{1}{3}$. As in Example 2.4, we have that $\overline{m_1}$ and $\overline{m_2}$ are σ-sup-decomposable, but they are not order continuous.

References

[1] Dobrakov, I.: On submeasures I, Dissertationes Math. 112, Waszawa, 1974.

[2] Dobrakov, I.: On extension of submeasures, Math. Slovaca 34, 1984, 265-271.

[3] Drewnowski, L.: Topological rings of sets, continuous set function, integration II, Bull. Acad. Polon. Sci. 20 (1972), 277-286.

[4] Drewnowski, L.: On the continuity of certain non-additive set functoins, Colloquium Math. 38 (1978), 243 - 253.

[5] Pap, E.: Lebesgue and Saks decompositions of $\perp$-decomposable measures, Fuzzy Sets and Systems. 38(1990), 345-353.

[6] Pap, E.: On non-additive set functions, Atti Sem. Mat. Fis. Universitá Modena 39(1991), 35-360.

[7] Squillante, M., Ventre, A., Weber, S.: Closure and convergence properties for classes of decomposable measures, Riv. di Math. Universitá Parma (to appear).

[8] Weber, S.: $\perp$-decomposable measures and integrals for Archimedean t-conorms, J. Math. Anal. Appl. 101, (1984), 114-138.

REZIME

PROŠIRENJE NEPREKIDNO t-CONORMA DEKOMPOZABILNE MERE

U radu se ispituju proširenja $\perp$-dekompozabilne mere (u odnosu na neprekidnu t-konormu $\perp$) sa prstena skupova $\mathcal{R}$ na σ-prsten Σ generisan sa $\mathcal{R}$ i na klasu $\mathcal{R}_\sigma$ koja se sastoji od skupova koji su granice rastućih nizova skupova iz $\mathcal{R}$.

Received by the editors February 12, 1990.