Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 33-38

Review of Research
Faculty of Science
Mathematics Series

SOME COMMUTATIVITY THEOREMS FOR RINGS VIA ITERATION

H.A.S. Abujabal and and **M.S. Khan**
Department of Mathematics, King Abdulazis University,
P.O. Box 31464 Jeddah-21497, Saudi Arabia

Abstract

Let m and n be fixed positive integers and R a ring with identity in which $x^m y^n = y^n x^m$ and $x^m y^{n+2} = y^{n+2} x^m$ hold for all $x, y \in R$. Then R is commutative provided it contains no non-zero elements x for which $p!x = 0$, where $p = \max\{m, n+1\}$. Another commutativity theorem for R under a different set of conditions is also obtained. The method of proof is based on an iteration technique.

AMS Mathematics Subject Classification (1980): Primary 16A70.
Key words and phrases: Ring, commutativity.

1. Introduction

Many sufficient conditions are well-known under which a given ring becomes commutative. Notable among them are some of those obtained by Jacobson, Kaplansky, Herstein, Bell, Kobayashi, Aby-Khuzam, Yagub, Tominaga, Hirano and several others. There are many commutativity theorems for rings satisfying polynomial identities involving consecutive integral powers of elements. The work of Abu-Khuzam [1], Abu-Khuzam - Yagub [2], Bell [3], Dong [4], Ligh - Richoux [5] and Tong [6] are worth mentioning.

In this note, we intend to prove some new commutativity theorems for a ring with identity in which a certain power of an element commutes with other elements having adjacent integral powers. We have used an iteration type method in the proof of these results.

The following results will be frequently used in the sequel. Throughout the rest of the paper, 1 stands for the indentity of a ring R.

Lemma 1.1 *(Bell [3]). Let x and y be elements of the ring R with 1. If there exists an integer $n \in Z^+$ for which $x^n y = (x+1)^n y = 0$, then $y = 0$.*

Lemma 1.2 *(Tong [6]). Let R be a ring with 1. Let $I_0^r(x) = x^r$. If $k > 1$, let $I_k^r(x) = I_{k-1}^r(1+x) - I_{k-1}^r(x)$. Then $I_{r-1}^r(x) = \frac{1}{2}(r-1)r! + r!x$; $I_r^r(x) = r!$, and $I_j^r(x) = 0$ for $j > r$.*

2. Results

The following is the main result of this paper.

Theorem 2.1 *Let m and n be fixed positive integers and R a ring with identity satisfyng the conditions*

$$x^m y^n = y^n x^m \text{ and } x^m y^{n+2} = y^{n+2} x^m,$$

for all x, y in R. Then R must be commutative provided it contains no non-zero elements x for which $p!x = 0$, where $p = \max\{m, n+1\}$.

Proof. Firstly, we note that the relations

$$x^m y^{n+2} = y^{n+2} x^2 = y^2(y^n x^m) = y^2(x^m y^n)$$

imply that

(1) $$[x^m, y^2] y^n = 0$$

for all x, y in R. Let $I_j(x) = I_j^m(x)$, $j = 0, 1, 2, 3, \ldots$ Then (1) can be written as

$$[I_0(x), y^2] y^n = 0.$$

Replacing x by $x+1$ in the above expression, we get

(2) $$[I_0(x+1), y^2] y^n = 0.$$

In view of Lemma 1.2, now equation (2) is reduced to

$$[I_1(x), y^2]y^n = 0. \tag{3}$$

Again, putting $x = x + 1$ in the above expression, we get

$$[I_1(x+1), y^2]y^2 = 0$$

for all x, y in R. Once again, an application of Lemma 1.2 yields for all x, y in R, the identity

$$[I_2(x), y^2]y^n = 0.$$

Finally, replacing x by $x + 1$ and iterating n times, we obtain

$$[I_{m-1}(x), y^2]y^n = 0$$

for all x, y in R. But, from Lemma 1.2, we known that $[I_{m-1}(x)] = \frac{1}{2}(m-1)m! + m!x$. So we are left with

$$m![x, y^2]y^n = 0.$$

Now, we shall iterate the power of y. For this, as above, we let $I_j(y) = I_j^n(y),\ \ j = 0, 1, 2, \ldots$, so we get

$$m![x, y^2]I_0(y) = 0 \tag{4}$$

for all x, y in R. Let us replace y by $y + 1$ and then get from the above expression the identity.

$$m![x, 1 + 2y + y^2]I_0(y + 1) = 0.$$

Using Lemma 1.2, we obtain

$$m!\{2[x, y](I_1(y) + I_0(y)) + [x, y^2](I_1(y) + I_0(y))\} = 0.$$

Then, in view of (4), we get

$$m!\{2[x, y]I_1(y) + 2[x, y]I_0(y) + [x, y^2]I_1(y)\} = 0.$$

Putting $y = y + 1$ again in the above expression, we have

$$m!\{2[x, y]I_1(y + 1) + 2[x, y]I_0(y + 1) + [x, (y + 1)^2]I_1(y + 1)\} = 0.$$

Then, as above after simplication, we get

$$m!\{4[x,y](I_2(y)+4[x,y]I_1(y)+[x,y^2]I_2(y)\}=0.$$

Replacing y by $y+1$, in the above relation we get

$$m!\{6[x,y]I_3(y)+6[x,y]I_2(y)+[x,y^2](I_3(y)\}=0.$$

Finally, putting $y=y+1$ in the last expression and iterating n times, we obtain

$$m!\{2(n+1)[x,y]I_{n+1}(y)+2(n+1)[x,y]I_n(y)+[x,y^2]I_{n+1}(y)\}=0.$$

But Lemma 1.2 suggests that $I_{n+1}(y)=0$ and $I_n(y)=n!$ giving thereby

$$2m!(n+1)![x,y]=0$$

for all x,y in R. Now, from the hypothesis of the theorem, the commutativity of R follows immediately. This completes the proof.

Remark. For $m>n$, $p=m$. But we can always suppose that $m>n$, since for $m=1$ (res. $m>1$), the first assumption of Theorem 2.1 is fulfilled for $n+1$ (resp. m^k, $k=1,2,3,\dots$) instead of m.

The next result provides another criterion for a ring to be commutative.

Theorem 2.2 *Let R be a ring with identity and suppose that a positive integer m exists such that for any x and y in R there is a positive integer $n=n(x,y)$ with $x^m y^n=y^n x^m$ and $x^m y^{n+1}=y^{n+1}x^m$. Then, R is necessarily commutative, whenever $m!x\neq 0$ except for $x=0$.*

Proof. We have

$$y^{n+1}x^m=y^{n+1}=(x^m y^n)y=(y^n x^m)y,$$

for all x,y in R. This implies

(1) $$y^n[y,x^m]=0$$

for all x,y in R. Now, replacing y by $(y+1)$ similarly we get

(2) $$(y+1)^{n'}[y,x^m]=0$$

for all x, y in R, where $n' = n(x, y+1)$. Then, Lemma 1.1 yields

$$[y, x^m] = 0 \tag{3}$$

for all x, y in R. Now we shall apply the iteration method.

Let $I_j(x) = I_j^m(x)$, for $j = 0, 1, 2, \ldots$
Then (3) can be re-written as

$$[y, I_0(x)] = 0 \tag{4}$$

for all x, y in R. Now, replacing x by $x+1$ in (4), we obtain for all x, y in R, the identity

$$[y, I_0(x+1)] = [y, I_1(x)] = 0 \tag{5}$$

Again, replacing x by $x+1$, (5) gives

$$[y, I_2(x)] = 0 \tag{6}$$

for all x, y in R.

Finally, after putting $x = x+1$ in (6) and iterating (m_1) times, we get

$$[y, I_{m-1}(x)] = 0. \tag{7}$$

But, from Lemma 1.2, $I_{m-1}(x) = \frac{1}{2}(m-1)m! + m!x$, so (7) is reduced to

$$m![y, x] = 0,$$

which, along with the hypothesis $m!x \neq 0$ except for $x = 0$, forces R to be commutative. This ends the proof.

Remarks.

i. In view of (3), our Theorem 2.2 can be derived from Theorem 3 of Tong [6]. However, we have include the proof of Theorem 2.2 just for the sake of completeness as Theorem 3 in [6] was stated without proof.

ii. A result similar to our Theorem 2.2 was obtained by Dong [4] for a semi prime ring. Our result is, however, proved under less stringent conditions on the ring.

iii. Only from (3) of the proof of Theorem 2.2, one can not conclude that R is necessarily commutative. To support this, we quote the famous example of the Corbas $(2, 2, \phi)$ - ring due to Bell [3]. In fact, one needs some torsion condition on the integer occuring as the power in (3) in order to establish the commutativity of R.

Acknowledgement

We are greatful to Professor H. E. Bell and Professor H. Abu-Khuzam for providing us with their reprints which have motivated the present study. Thanks is also due to the learned referee for most valuable comments on the earlier version of our paper.

References

[1] Abu-Khuzam, H.: n-torsion free rings with commuting powers, Math. Japon., 25 (1980), 37-42.

[2] Abu-Khuzam, H., Adil Yagub: Rings and groups with commuting powers, Internat. J. Math. and Math. Sci., 4 (1981), 101-107.

[3] Bell, H.E.: On rings with commuting powers, Math. Japon., 24 (1979), 473-478.

[4] Dong, N.C.: Commutativity of some non-associative rings with identity, Acta Sci. Natur. Univ. Jilin, 1 (1983), 1-9.

[5] Ligh, S., Richuox, A.: A commutativity theorem for rings, Bull. Austral. Math. Soc., 16 (1977), 75-77.

[6] Tong, J.: On the commutativity of a ring with identity, Canad. Math. Bull., 27 (4) (1984), 456-460.

REZIME

NEKE TEOREME O KOMUTATIVNOSTI PRSTENA PREKO ITERACIJE

Neka su m i n fiksni pozitivni celi brojevi i R prsten sa jedinicom u kome $x^m y^n = y^n x^m$ and $x^m y^{n+2} = y^{n+2} x^m$ važi za sve x, y. Tada je R komutativan ako ne sadrži ni jedan ne-nula element x za koji $p!x = 0$, gde je $p = \max\{m, n+1\}$. Druga teorema o komutativnosti za R je takodje dobijena, pod drugim uslovima. Metod dokazivanja je zasnovan na jednoj iteracionoj tehnici.

Recived by the editors August 8, 1989.