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CONNECTEDNESS OF THE GENERALIZED DIRECT PRODUCT OF REGULAR DIGRAPHS

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Abstract

Using the spectral method a theorem is proved giving the necessary and sufficient conditions for the generalized direct product (GDP) of regular (di)graphs to be connected (di)graph.

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1. Introduction

Let B be a set of n-tuples $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ of symbols 1,0,-1 which does not contain an n-tuple (0,0,...,0). If G is a digraph with at most ν parallel arcs (if there are no parallel arcs then $\nu = 1$) between any two vertices or loops of a vertex in G, then complement $\bar{G}$ of G is a digraph which has the same set of vertices as G and for any ordered pair (u, v) of vertices u and v of $\bar{G}$ (if loops are not allowed then $u \neq v$) from u to v lead $\nu - a$ arcs, where a is the number of arcs leading from u to v in G.

The following definition is introduced in [5] and [6] and represent a generalization of the definition of the GDP of graphs [7] to digraphs (digraphs can have multiple arcs and or loops).

Definition 1. *The generalized direct product with a basis B of digraphs G_1, G_2 ,*

$\ldots, G_n$ is the digraph G=GDP $(B; G_1, \ldots, G_n)$ whose set of vertices is the Cartesian product of the sets of vertices of digraphs $G_1, G_2, \ldots, G_n$.For two vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ of G construct all the possible arc selections of the following type. For each $(\beta_1, \ldots, \beta_n)$ from B, for which $u_k = v_k$ holds whenever $\beta_k = 0$, and for any i ($i = 1, 2, \ldots, n; i \neq k$) select an arc going from u_i to v_i in G_i if $\beta_i = 1$ and an arc going from u_i to v_i in $\bar{G}_i$

mboxif $\beta_i = -1$. The number of arcs going from u to v in G is equal to the number of such selections.□

If B consists of n-tuples of symbols 1 and 0 the resulting operation is called a non-complete extended p-sum (NEPS). Conditions, under which this operation on arbitrary digraphs is a strongly connected digraph, have been investigated in [3].

It is clear that the GDP $(B; G_1, \ldots, G_n)$ can be connected if any one of the digraphs $G_1, \ldots, G_n$ is not connected. On the other hand, if $G_i (i = 1, \ldots, n)$ has at least two vertices, then the GDP $(B; G_1, \ldots, G_n)$ is not connected if basis B has not the property (D): that for every $j \in \{1, \ldots, n\}$ there exist in B at least one n-tuple $(\beta_1, \ldots, \beta_n)$ with $\beta_j \neq 0$. (This cond

ition implies that the GDP, effectively, depends on each G_i.) Further, if $G_j(\bar{G}_j)$, for some $j \in \{1, \ldots, n\}$, is not connected, then the GDP $(B; G_1, \ldots, G_n)$ is not connected, if each n-tuple $(\beta_1, \ldots, \beta_n)$ in B has $\beta_j \neq -1$ $(\beta_j \neq 1)$.

In order to get conditions for $B, G_1, \ldots, G_n$ being GDP $(B; G_1, \ldots, G_n)$ strongly connected, we shall investigate the connectivity of the isomorphic GDP given from this one by the replacement of each non-connected digraph G_j by its complement $\bar{G}_j$ together with the replacement in each $\beta \in B$ j-th coordinate 1 by -1 and vice versa (Proposition 3 in [5]). So we suppose that in GDP $(B; G_1, \ldots, G_n)$ all the $G_1, \ldots, G_n$ are strongly connected. In the same manner, we shall consider the GDP whose basis B has property (E): if for any $j \in \{1, \ldots, n\}$ all β_j (j-th coordinate of $\beta \in B$) take only one value among 1 and -1, then $\beta_j \in \{1, 0\}$.In this case if G_j is not connected, then GDP $(B; G_1, \ldots, G_n)$ is not connected.

A digraph is called a regular of degree r if each indegree and each out-degree equals r. The cycle, denoted by $\vec{C}_p$, is a connected regular digraph of degree 1 with p verices. Digraph G is called bicomplete if G is a complete bipartite graph (symmetric). Notice that the strong components of a regular digraph are its components too.

We shall investigate the connectedness of the GDP of regular digraphs by using Theorems 0.3 and 0.4 from [2].

For this purpose we need the following results of [5].

Theorem 1. *([5]). The GDP of regular digraphs is a regular digraph.*

Theorem 2. *([5]). Let G be a regular digraph of order p, degree r and maximum number of parallel arcs or loops of a vertex equal to ν and let $S = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$ be the spectrum of G. The complement $\bar{G}$ of the digraph G has the spectrum given by*
$S_1 = \{\bar{\lambda}_1 = \nu \cdot p - r, \bar{\lambda}_2 = -\lambda_2, \ldots, \bar{\lambda}_p = -\lambda_p\}$, if loops are allowed,
$S_2 = \{\bar{\lambda}_1 = \nu \cdot p - \nu - r, \bar{\lambda}_2 = -\nu - \lambda_2, \ldots, \bar{\lambda}_p = -\nu - \lambda_p\}$, if loops are not allowed.

The eigenvectors correspoding to λ_i and $\bar{\lambda}_i$ are the same and the eigenvector belonging to the eigenvalue λ distinct from r in G is orthogonal to the eigenvector (1,...,1) belonging to the r. □

Theorem 3. *([5]). For i=1,2,...,n let G_i be a regular digraph with p_i vertices, degree r_i and let λ_{ij_i} $(\bar{\lambda}_{ij_i})$ $j_i = 1, 2, \ldots, p_i$ be the spectrum of $G_i(\bar{G}_i$ determined by Theorem 2). Then, the spectrum of the GDP $(B; G_1, \ldots, G_n)$ consists of all the possible values of $\Lambda_{j_1,\ldots,j_n}$ where*

$$\Lambda_{j_1,\ldots,j_n} = \sum_{\beta \in B} \lambda_{1_{j_1}}^{[\beta_1]} \lambda_{2_{j_2}}^{[\beta_2]} \ldots \lambda_{n_{j_n}}^{[\beta_n]}, \tag{1}$$

$$\lambda_{i_{j_i}}^{[1]} = \lambda_{i_{j_i}}, \lambda_{i_{j_i}}^{[0]} = 1, \lambda_{i_{j_i}}^{[-1]} = \bar{\lambda}_{i_{j_i}}, j_i = 1, 2, \ldots, p_i;\ i = 1, 2, \ldots, n$$

The eigenvector $x_{j_1, \ldots, j_n} = x_{1_{j_1}} \otimes x_{2_{j_2}} \otimes \ldots \otimes x_{n_{j_n}}$ corresponds to the eigenvalue $\Lambda_{j_1, \ldots, j_n}$, where $x_{i_{j_i}}$ is the eigenvector belonging to $\lambda_{i_{j_i}}$ in G_i and $\otimes$ denotes the Kronecker product of matrices. □

2. Main theorems

Let h be the greatest common divisor of the lengths of all the cycles in a digraph G. The digraph G is called primitive if it is strongly connected and $h = 1$ [4,p.210], and imprimitive if it is strongly connected and $h > 1$. In the second case, h is called the index of imprimitivity (h is the index of imprimitivity of the adjacency matrix of the digraph G as well [1,p.183].

Theorem 4. *Let $G_i, i = 1, 2, \ldots, n$ be a regular, connected, non-complete symmetric digraph of degree r_i containing $p_i (p_i \geq 2)$ vertices. Suppose also that loops are not allowed in G_i and $\bar{G}_i$ $(i = 1, 2, \ldots, n)$. Further, let digraphs $G_{i_1}, G_{i_2}, \ldots, G_{i_s}(\{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, n\})$ be imprimitive with the imprimitivity indices $h_{i_1}, h_{i_2}, \ldots, h_{i_s}$ respectively. The GDP with the basis B satisfying conditions (D) and (E) of digraphs $G_1, G_2, \ldots, G_n$ is a connected digraph if and only if for every non-empty subset $\{j_1, j_2, \ldots, j_k\}$ of $\{i_1, i_2, \ldots, i_s\}$ and every choice of integers $\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_k}, 1 \leq \ell_{j_t} \leq h_{j_t} - 1, t = 1, 2, \ldots, k$ there exist $\beta \in B$ such that $\{j_1, j_2, \ldots, j_k\} \cap \{\nu \mid \beta_\nu \neq 0\} = I_\beta \neq 0$ and the following holds: for at least one $\nu \in I_\beta$, for which $\beta_\nu = -1, G_\nu$ is neither bicomplete nor $\vec{C}_3$ or otherwise*

$$\sum_{\nu \in I_\beta} \left(\frac{1}{2}(1 + \beta_\nu)\frac{\ell_\nu}{h_\nu} + \frac{\ell_\nu}{3}(1 - \beta_\nu)(p_\nu - 2r_\nu)\right)$$

is not an integer.

Proof. According to Theorems 0.3 and 0.4 from [3] a digraph G, with an adjecency matrix A, is strongly connected if and only if its index r is a simple eigenvalue and if the positive eigenvectors belong to r both in A and A^T.

Without loss of generality we may suppose that the digraphs $G_1, G_2, \ldots, G_n$ are without multiple arcs.

By Theorem 3 the index of G=GDP $(B; G_1, \ldots, G_n)$ is $\Lambda = \sum_{\beta \in B} r_1^{[\beta_1]} r_2^{[\beta_2]} \ldots r_n^{[\beta_n]}$ and $(1, 1, \ldots, 1)$ is the positive eigenvector belonging to the index Λ in the adjacency matrix A of G and A^T, where $r_i^{[1]} = r_i, r_i^{[0]} = 1$ and $r_i^{[-1]} = p_i - r_i - 1$.

By the same theorem, if none of the G_i is complete, the index of GDP can be obtained only from those eigenvalues of the digraphs $G_i(\bar{G}_i), i =$

$1, 2, \ldots, n$ which have a modulus equal to $r_i(p_i - r_i - 1)$. All these eigenvalues of G_j can be written in the form $r_j \exp(l_j \frac{2\pi}{h_j})$, $0 \le l_j \le h_j - 1$ ($\exp(t) = e^{ti}$, $i^2 = -1$) (Theorem of Frobenius). Therefore, from (1) we have

$$\text{(2)} \qquad \Lambda = \sum_{\beta \in B} \prod_{i=1}^{n} \Big(\frac{1}{2}(\beta_i^2 + \beta_i) r_i \exp(\frac{\ell_i}{h_i} 2\pi) + (1 - \beta_i^2) +$$

$$+ \frac{1}{2}(\beta_i^2 - \beta_i)(\bar{sg}(\ell_i) p_i - 1 - r_i \exp(\frac{l_i}{h_i} 2\pi))\Big),$$

where $\bar{sg}(0) = 1$ and $\bar{sg}(x) = 0$ for $x > 0$. From (2) it follows that the index Λ is a simple eigenvalue if for each choice of integers $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_s}, 0 \le \ell_{i_t} \le h_{i_t} - 1 (t = 1, 2, \ldots, s)$ with at least one $\ell_{i_t} > 0$, at least one summand in Λ is differen from $r_1^{[\beta_1]}, r_2^{[\beta_2]} \ldots r_n^{[\beta_n]}$.

For any choice of integers $\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_k}, 1 \le \ell_{j_t} \le h_{j_t} - 1, \{j_1, j_2, \ldots, j_k\} \subset \{i_1, i_2, \ldots, i_s\}$ and any $\beta \in B$ let $I_\beta = \{j_1, j_2, \ldots, j_k\} \cap \{\nu \mid \beta_\nu \ne 0\}$. Then, from (2) we have ($I = \{1, 2, \ldots, n\}$):

$$\Lambda = \sum_{\beta \in B} \Big(\prod_{i \in I \setminus I_\beta} r_i^{[\beta_i]}\Big)\Big(\prod_{\nu \in I_\beta} \Big(\frac{1}{2}(1 + \beta_\nu) r_\nu \exp(\frac{\ell_\nu}{h_\nu} 2\pi) +$$

$$+ \frac{1}{2}(1 - \beta_\nu)(-1 - r_\nu \exp(\frac{\ell_\nu}{h_\nu} 2\pi))\Big)\Big),$$

or

$$\text{(3)} \quad \Lambda = \sum_{\beta \in B} \Big(\prod_{i \in I \setminus I_\beta} r_i^{[\beta_i]}\Big)\Big(\prod_{\nu \in I_\beta} (r_\nu^2 + \frac{1}{2}(1 - \beta_\nu)(2 r_\nu \cos \frac{\ell_\nu}{h_\nu} 2\pi + 1))^{\frac{1}{2}} \times$$

$$\times \exp\Big(\frac{1}{2}(1 + \beta_\nu)\frac{\ell_\nu}{h_\nu} 2\pi + \frac{1}{2}(1 - \beta_\nu)\Theta_\nu\Big)\Big).$$

Hence, index Λ is a simple eigenvalue of GDP if and only if for at least one $\beta \in B$ one of the following conditions is satisfied : (a) there exists $\nu \in I_\beta$ such that $\beta_\nu = -1$ and $(r_\nu^2 + 2 r_\nu \cos \frac{\ell_\nu}{h_\nu} 2\pi + 1)^{\frac{1}{2}} \ne p_\nu - r_\nu - 1$ or (b) the argument of the operator exp of the corresponding summand in Λ is different from $2k\pi, k \in Z$. From (a) the following two cases arise: $1^\circ p_\nu - r_\nu - 1 = (r_\nu^2 + 2 r_\nu \cos \frac{\ell_\nu}{h_\nu} 2\pi + 1)^{\frac{1}{2}} = r_\nu - 1$ and $2^\circ p_\nu - r_\nu - 1 = (r_\nu^2 + 2 r_\nu \cos \frac{\ell_\nu}{h_\nu} 2\pi + 1)^{\frac{1}{2}} = r_\nu$. The case $(r_\nu^2 + 2 r_\nu \cos \frac{\ell_\nu}{h_\nu} 2\pi + 1)^{\frac{1}{2}} = r_\nu + 1$ is impossible, because $\ell_\nu > 0$. If 1° holds then $\cos \frac{\ell_\nu}{h_\nu} 2\pi = -1$, consequently G_ν

is bicomplete ($p_\nu = 2r_\nu, h_\nu = 2l_\nu$) and $\Theta_\nu = 0$. If 2° holds, then $\cos \frac{\ell_\nu}{h_\nu} 2\pi = -\frac{1}{2r_\nu}$, which is possible only for $r_\nu = 1$, when $p_\nu = 2r_\nu + 1 = 3$ and $h_\nu = 3$ i.e. G_ν is $\vec{C}_3$. In this case $\Theta = \frac{4\pi}{3}$ if $\ell_\nu = 1$ and $\Theta_\nu = \frac{2\pi}{3}$ if $\ell_\nu = 2$. From these facts, the first part of the theorem follows.

Using condition (b) and supposing that 1° or 2° of (a) is satisfied for each $\nu \in I_\beta$ for which $\beta_\nu = -1$, we get the following condition

$$\sum_{\nu \in I_\beta} \left(\frac{1}{2}(1+\beta_\nu)\frac{\ell_\nu}{h_\nu}2\pi + \frac{1}{2}(1-\beta_\nu)\frac{4\pi}{3}\ell_\nu(p_\nu - 2r_\nu)\right) \neq 2k\pi,$$

which completes the proof of the theorem. □

Theorem 5. *Let $G_i, i = 1,2,\ldots,n$ be a regular, connected, non-complete symmetric digraph with at least two vertices. Suppose also that loops are permitted in the digtaphs. Further, let digraphs $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$ ($\{i_1, i_2, \ldots, i_s\} \subset \{1,2,\ldots,n\}$) be imprimitive with the imprimitivity indices $h_{i_1}, h_{i_2}, \ldots, h_{i_s}$ respectively. The GDP with the basis B satisfying conditions (D) and (E) of digraphs $G_1, G_2, \ldots, G_n$ is a connected digraph if and only if for every non-empty subset $\{j_1, j_2, \ldots, j_k\}$ of $\{i_1, i_2, \ldots, i_s\}$ and for every choice of integers $\ell_{j_1}, \ell_{j_2}, \ldots \ell_{j_k}, 1 \leq \ell_{j_t} \leq h_{j_t} - 1, t = 1,2,\ldots,k$ there exists $\beta \in B$, such that $\{j_1, j_2, \ldots, j_k\} \cap \{\nu \mid \beta_\nu \neq 0\} = I_\beta \neq \emptyset$ and the following holds: for at least one $\nu \in I_\beta$, for which $\beta_\nu = -1, G_\nu$ is not bicomplete or otherwise*

$$\sum_{\nu \in I_\beta} \left(\frac{\ell_\nu}{h_\nu} + \frac{1}{4}(1-\beta_\nu)\right)$$

is not an integer.

Proof. We use the same proof method as in Theorem 4.We also suppose that the digraphs $G_1, G_2, \ldots, G_n$ are without multiple arcs.

If $p_i(r_i)$ denotes the number of vertices (degres-index) of $G_i (i = 1,2,\ldots,n)$, then by Theorem 3 the index of $G = \mathrm{GDP}(B; G_1, \ldots, G_n)$ is $\Lambda = \sum_{\beta \in B} r_1^{[\beta_1]} r_2^{[\beta_2]} \ldots r_n^{[\beta_n]}$ and $(1,1,\ldots,1)$ is the positive eigenvector belonging to the index Λ in the adjacency matrix A of G and A^T, where $r_i^{[1]} = r_i, r_i^{[0]} = 1$ and $r_i^{[-1]} = p_i - r_i$.

Similarly as in the previous theorem index Λ of GDP $(B; G_1, \ldots, G_n)$ can be obtained from those eigenvalues of the digraphs $G_i(\bar{G}_i)$, $i = 1,2,\ldots,n$

which have a modulus equal to $r_i(p_i - r_i)$ i.e. from the exspresion

$$\Lambda = \sum_{\beta\in B} \prod_{i=1}^{n} (\frac{1}{2}(\beta_i^2 + \beta_i) r_i \exp(\frac{\ell_i}{h_i} 2\pi) + (1 - \beta_i^2) + \quad (4)$$

$$+ \frac{1}{2}(\beta_i^2 - \beta_i)(\bar{sg}(\ell_i) p_i - r_i \exp(\frac{\ell_i}{h_i} 2\pi))).$$

For any choice of integers $\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_k}$, $1 \le \ell_{j_t} \le h_{j_t} - 1$, $\{j_1 . j_2, \ldots, j_k\} \subset \{i_1, i_2, \ldots, i_s\}$ and any $\beta \in B$ let $\{j_1, j_2, \ldots, j_k\} \cap \{\nu | \beta_\nu \ne 0\} = I_\beta$. Then from (4) we have $(I = \{1, 2, \ldots, n\})$:

$$\Lambda = \sum_{\beta\in B} (\prod_{i\in I\setminus I_\beta} r_i^{[\beta_i]}) (\prod_{\nu\in I_\beta} (\frac{1}{2}(1 + \beta_\nu) r_\nu \exp(\frac{\ell_\nu}{h_\nu} 2\pi) +$$

$$+ \frac{1}{2}(1 - \beta_\nu)(-r_\nu \exp(\frac{\ell_\nu}{h_\nu} 2\pi))))$$

or

$$\Lambda = \sum_{\beta\in B} (\prod_{i\in I\setminus I_\beta} r_i^{[\beta_i]}) (\prod_{\nu\in I_\beta} r_\nu \exp(\frac{\ell_\nu}{h_\nu} 2\pi + \frac{1}{2}(1 - \beta_\nu)\pi)).$$

Hence, index Λ is a simple eigenvalue of GDP if and only if for at least one $\beta \in B$ one of the following conditions is satisfied: (a) there exists $\nu \in I_\beta$ such that $\beta_\nu = -1$ and $r_\nu \ne p_\nu - r_\nu$ i.e. G_ν is not bicomplete or (b) the argument of the operator exp of the corresponding summand in Λ is different from $2k\pi$, $k \in Z$. From these facts the statement of th

is theorem follows. □

The following theorem is a specialization of the previous once to undirected graphs.

Theorem 6. *Let* G_i, $i = 1, 2, \ldots, n$ *be a connected regular, non complete, graph with at least two vertices and let* $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$, $\{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, n\}$ *be bipartite. The GDP with the basis B satisfying conditions (D) and (E) of graphs* $G_1, G_2, \ldots, G_n$ *is a connected graph if and only if for every non-empty subset* $\{j_1, j_2, \ldots, j_k\}$ *of* $\{i_1, i_2, \ldots, i_s\}$, *there exists* $\beta \in B$ *such that* $\{j_1, j_2, \ldots, j_k\} \cap \{\nu | \beta_\nu \ne 0\} = I_\beta \ne \emptyset$ *and one of the following conditions is satisfied:*

(i) For at least one $\nu \in I_\beta$, $\beta_\nu = -1$ *and* G_ν *is not bicopmlete.*

(ii) The number of $\nu \in I_\beta$ *for which* $\beta_\nu = 1$ *is odd.* □

This theorem holds that either loops are or are not allowed in graphs.

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REZIME

POVEZANOST GENERALISANOG DIREKTNOG PROIZVODA REGULARNIH DIGRAFOVA

Korištenjem spektralnog metoda dobijeni su potrebni i dovoljni uslovi (teoreme 4 i 5) da generalisani direktan proizvod (GDP) regularnih digrafova bude povezan digraf.

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