Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 21, 1 (1991), 75-81

Review of Research
Faculty of Science
Mathematics Series

THE BROOKS-JEWETT THEOREM FOR NON-ADDITIVE SET FUNCTIONS

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Abstract

In this paper a version of the Brooks-Jewett theorem on the convergence of sequences of set functions which have ranges in an arbitrary uniform space (without considering any algebraic operation on it) is proved.

AMS Mathematics Subject Classification (1980): 28B99
Key words and phrases: Uniform space, x_0–exhaustive set function

The Brooks-Jewett theorem [1], as an additive version of the Nikodym convergence theorem, was considered and generalized also for non-additive set functions [3], [4], [5], [6], [7], [9], [10], [11], [12]. In all these papers the ranges of the considered set functions (additive, subadditive, k–subadditive, k–triangular, etc.) were endowed with some algebraic operation.

In our paper [11] we have proved a Nikodym uniform boundedness type theorem for set functions with the values in an arbitrary uniform space. Using the approach from paper [11] we shall obtain a version of the Brooks-Jewett theorem for set functions defined on a quasi-σ-ring and with the values in a uniform space.

Let Y be a uniform space endowed with the uniformity $\mathcal{U}$. We shall use the following definition of the boundedness given by J. Hejcman [8].

Definition 1. *A subset B of Y is bounded ($\mathcal{U}$-bounded) if for every $U \in \mathcal{U}$ there exist a finite set $K \subset B$ and a natural number n such that*

$$B \subset U^n[K],$$

where $U^1 = U$, $U^n = U \circ U^{n-1}$ ($\circ$ is the composition of the relations) and $U[K]$ is the set of all $x \in Y$ such that $(x, y) \in U$ for some $y \in K$.

We shall use following characterization of the $\mathcal{U}$-boundedness (Theorem 1.12 from [8]).

Theorem 1. *A set $B \subset Y$ is $\mathcal{U}$-bounded if and only if it is $d-$bounded for every uniformly continuous pseudo-metric d defined on Y.*

Using this theorem we shall denote with $\mathcal{D}$ the family of all the uniformly continuous pseudometrics defined on $(Y,\mathcal{U})$.

Definition 2. *A ring of sets $\sum$ is called a quasi-σ-ring if any disjoint sequence in $\sum$ posseses a subsequence which belongs to the family of disjoint sequences $\{A_n\}$ in $\sum$ for which*

$$\left\{\bigcup_{n\in M} A_n : M \subset \mathbf{N}\right\} \subset \sum .$$

In the whole paper $\sum$ will always denote a quasi-σ-ring.

Definition 3. *A set function $\mu : \sum \to Y$ is said to be x_0-exhaustive, for $x_0 \in Y$, if for each $d \in \mathcal{D}$*

$$\lim_{n\to\infty} d(\mu(E_n),\ x_0) = 0$$

for each sequence $\{E_n\}$ of pairwise disjoint sets from $\sum$.

For $d \in \mathcal{D}$ the <u>d – semivariation</u> ([11]) of the set function μ, $\mu : \sum \to Y$, with respect to a point $x_0 \in Y$ is

$$\widetilde{\mu}_d^{x_0}(B) := \sup\{d(\mu(C), x_0) : C \subset B,\ C \in \textstyle\sum\}\ \ (B \in \textstyle\sum).$$

We shall need following (Lemma 2.2.2 from [11])

Lemma 1. *Let $\mu : \sum \to Y$ be an x_0-exhaustive set function, and let $\{A_n\}$ be a sequence of pairwise disjoint elements from $\sum$. Then, for each $d \in \mathcal{D}$ and each $\varepsilon > 0$, there exists a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that*

$$\widetilde{\mu}_d^{x_0}\left(\bigcup_{i \in I} A_{n_i}\right) < \varepsilon,$$

for any $I \subset \mathbf{N}$.

We introduced in [11] for $d \in \mathcal{D}$ and $x_0 \in Y$

$$\alpha_d^{x_0}(A, \mu) := \lim_{n \to \infty} \sup\{d(\mu(A \cup B), x_0) : \widetilde{\mu}_d^{x_0}(B) < \frac{1}{n},\ B \in \sum\}$$

$$(A \in \sum,\ \mu : \sum \to Y).$$

Now we have the Brooks-Jewett type theorem.

Theorem 2. *Let $\{\mu_n\}$ be a sequence of set functions μ_n, $\mu_n : \sum \to Y$, such that they satisfy the following conditions for an arbitrary but fixed $x_0 \in Y$*

(i) for each $d \in \mathcal{D}$ and for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\mu_n(A), x_0) < \delta$ and $d(\mu_n(B), x_0) < \delta$ for $B \subset A$, $A, B \in \sum$, $n \in \mathbf{N}$ implies

$$d(\mu_n(A \backslash B), x_0) < \varepsilon,$$

(ii) for each $d \in \mathcal{D}$ and for each $\delta > 0$, there exists $\Theta > 0$ such that $d(\mu_n(A), x_0) < \Theta$, $A \in \sum$, $n \in \mathbf{N}$ implies

$$\alpha_d^{x_0}(A, \mu_n) < \delta, \quad n \in \mathbf{N},$$

(iii) for each $d \in \mathcal{D}$

$$\lim_{n \to \infty} d(\mu_n(E),\ \mu(E)) = 0$$

for each $E \in \sum$.

Then μ is x_0-exhaustive if and only if μ_n, $n \in \mathbf{N}$, are uniformly x_0-exhaustive.

Proof. Suppose that μ is x_0–exhaustive,but the sequence $\{\mu_n\}$ is not uniformly x_0–exhaustive. Then there exists $\varepsilon > 0$, d from $\mathcal{D}$ and a sequence $\{E_k\}$ of pairwise disjoint sets from $\sum$ and a subsequence $\{\mu_{n_k}\}$ such that

$$(1) \qquad d(\mu_{n_k}(E_k), x_0) > \varepsilon.$$

We choose $\delta > 0$ by (i) corresponding to $\varepsilon > 0$. Now we choose $\Theta > 0$ by (ii) corresponding to $\delta > 0$. By the x_0–exhaustivity of μ and the Lemma 1 there exists a subsequence $\{E_{k_i}\}$ of the sequence $\{E_k\}$ such that

$$(2) \qquad \tilde{\mu}_d^{x_0}\left(\bigcup_{i\in I} E_{k_i}\right) < \frac{\Theta}{2}$$

for any $I \subset \mathbf{N}$. Now, we shall take $m_i = \mu_{n_{k_i}}$ and $A_i = E_{k_i}$ $(i \in \mathbf{N})$ and $i_1 = 1$. Since we have

$$d(m_i(A_{i_1}), \mu(A_{i_1})) \geq d(m_i(A_{i_1}), x_0) - d(\mu(A_{i_1}), x_0)$$

and by (iii) there exists an index i_2 such that

$$d(m_{i_2}(A_{i_1}), \mu(A_{i_1})) < \frac{\Theta}{2},$$

using also (2) we obtain

$$(3) \qquad d(m_{i_2}(A_{i_1}), x_0) < \Theta.$$

By the x_0–exhaustivity of m_{i_2} we have by the Lemma that there exists a subsequence $\{A_i^2\}$ of the sequence $\{A_i\}_{i=i_1+1}^{\infty}$ such that

$$(\tilde{m}_{i_2})_d^{x_0}\left(\bigcup_{i\in I} A_i^2\right) < \frac{\Theta}{2}.$$

Hence by (3) and (ii), we have

$$(4) \qquad \alpha_d^{x_0}\left(A_{i_1} \bigcup \bigcup_{i\in I} A_i^2, m_{i_2}\right) < \delta$$

for any $I \subset \mathbf{N}$. By (2) $d(\mu(A_{i_1} \cup A_k^2), x_0) < \frac{\Theta}{2}$ for each $k \in \mathbf{N}$ and by (iii) there exists an index i_3 such that

$$d(m_{i_3}(A_{i_1} \cup A_k^2),\ \mu(A_{i_1} \cup A_k^2)) < \frac{\Theta}{2}.$$

Then, the inequality

$$d(m_i(A_{i_1} \cup A_k^2), \mu(A_{i_1} \cup A_k^2)) \geq d(m_i(A_{i_1} \cup A_k^2), x_0) - d(\mu(A_{i_1} \cup A_k^2), x_0)$$

with the preceding two facts implies

$$d(m_{i_3}(A_{i_1} \cup A_{i_2}), x_0) < \Theta, \tag{5}$$

where A_{i_2} is taken from the sequence $\{A_k^2\}$.

By x_0–exhaustivity of m_{i_3}, we have by the Lemma 1 that there exists a subsequence $\{A_i^3\}$ of the sequence $\{A_i^2\}_{i=i_2+1}^{\infty}$ such that

$$(\widetilde{m}_{i_3})_d^{x_0}\left(\bigcup_{i \in I} A_i^3\right) < \frac{\Theta}{2}$$

for any $I \subset \mathbf{N}$. Hence by (5) and (ii), we have

$$d(m_{i_3}\left(A_{i_1} \bigcup A_{i_2} \bigcup \bigcup_{i \in I} A_i^3\right), x_0) < \delta$$

for any $I \subset \mathbf{N}$.

Continuing this procedure we obtain two sequences $\{m_{i_k}\}$ and $\{A_{i_k}\}$. If we take $A_0 = \bigcup_{k=1}^{\infty} A_{i_k}$, then by (iii) there exists an index k_0 such that

$$d(m_{i_{k_0}}(A_0), x_0) < \eta < \delta. \tag{6}$$

Namely, this follows by (2) and the inequality

$$d(m_{i_{k_0}}(A_0), \mu(A_0)) \geq d(m_{i_{k_0}}(A_0), x_0) - d(\mu(A_0), x_0).$$

From the procedure of the whole previous construction, it follows that

$$d(m_{i_{k_0}}(A_0 \backslash A_{i_{k_0}}), x_0) < \delta.$$

Hence, using (i), we have by (6)

$$\varepsilon > d(m_{i_{k_0}}(A_0 \backslash (A_0 \backslash A_{i_{k_0}}), x_0) = d(m_{i_{k_0}}(A_{i_{k_0}}), x_0),$$

which is in contradiction with (1).

Now, if we suppose that $\mu_n (n \in \mathbf{N})$ are uniformly x_0–exhaustive, then by (iii) it follows that μ is x_0–exhaustive.

Corollary 1. *(Theorem 1. from [9]) Let $\{\mu_n\}$ be a sequence of $k-$triangular exhaustive set functions $\mu_n : \sum \to R^+$. If there exists*

$$\lim_{n\to\infty} \mu_n(E) = \mu(E)$$

for each $E \in \sum$ and μ is exhaustive, then $\{\mu_n\}$ is uniformly exhaustive and μ is $k-$triangular.

Proof. Conditions (i) and (ii) follow by $k-$triangularity.

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REZIME

BROOKS-JEWETTOVA TEOREMA ZA NEADITIVNE SKUPOVNE FUNKCIJE

U radu se dokazuje teorema tipa Brooksa-Jewetta o konvergenciji niza skupovnih funkcija sa vrednostima u proizvoljnom uniformnom prostoru (u opštem slučaju bez algebarske operacije). U tu svrhu se koriste rezultati iz ranijeg rada [11].

Received by the editors September 20, 1990.