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FIXED POINTS ON TWO COMPLETE METRIC SPACES

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Abstract

Let (X, d) and (Y, e) be two complete metric spaces. It is proved that if T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalites

$$e^2(Tx, TSy) \leq$$

$$c_1 \max\{d(x, Sy)e(y, Tx), d(x, Sy)e(y, TSy), e(y, Tx)e(y, TSy)\}.$$

$$d^2(Sy, STx) \leq$$

$$c_2 \max\{e(y, Tx)d(x, Sy), e(y, Tx)d(x, STx), d(x, Sy)d(x, STx)\}.$$

for all x in X and y in Y, where $0 \leq c_1 \cdot c_2 < 1$ or the inequalities

$$e(Tx, TSy) \cdot \max\{e(y, Tx), e(TSy, y)\}$$

$$\leq c_1 d(x, Sy) \cdot \max\{d(x, Sy), e(y, TSy)\}$$

$$d(Sy, STx) \cdot \max\{d(x, Sy), d(x, STx)\} \leq$$

$$\leq c_2 e(y, Tx) \cdot \max\{e(y, Tx), d(x, STx)\}$$

for all x in X and y in Y, where $0 \leq c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

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In a recent paper [1], the following theorem was proved

Theorem 1. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$e(Tx,TSy) \leq c\ \max\{d(x,Sy),e(y,Tx),e(y,TSy)\} \tag{1}$$

$$d(Sy,STx) \leq c\ \max\{e(y,Tx),d(x,Sy),d(x,STx)\} \tag{2}$$

for all x in X and y in Y, where $0 \leq c < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

In [2] and [3] the other fixed point theorems on two metric spaces are proved. Now, we shall prove two fixed point theorems involving two metric spaces.

Theorem 2. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$e^2(Tx,TSy) \leq \tag{3}$$

$$c_1 \max\{d(x,Sy)e(y,Tx),d(x,Sy)e(y,TSy),e(y,Tx)e(y,TSy)\}$$

$$d^2(Sy,STx) \leq \tag{4}$$

$$c_2 \max\{e(y,Tx)d(x,Sy),e(y,Tx)d(x,STx),d(x,Sy)d(x,STx)\}$$

for all x in X and y in Y, where $0 \leq c_1 \cdot c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. Let x be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$(ST)^n x = x_n,\ \ T(ST)^{n-1}x = y_n$$

for $n = 1,2,\ldots$. Using inequality (4) we have

$$d^2(x_n,x_{n+1}) \leq$$

$$\leq c_2 \max\{e(y_{n+1}, y_n)d(x_n, x_n), d(x_n, x_n)d(x_n, x_{n+1}),$$

$$e(y_n, y_{n+1})d(x_n, x_{n+1})\} = c_2 e(y_n, y_{n+1})d(x_n, x_{n+1}),$$

which implies

$$d(x_n, x_{n+1}) \leq c_2\, e(y_n, y_{n+1})$$

if $d(x_n, x_{n+1}) \neq 0$ and by using inequality (3), we have

$$e^2(y_n, y_{n+1}) \leq$$

$$c_1 \max\{d(x_{n-1}, x_n)e(y_n, y_{n+1}), e(y_n, y_n)e(y_{n+1}, y_n), d(x_{n-1}, x_n)e(y_n, y_{n+1})\} =$$

$$c_1 d(x_{n-1}, x_n)e(y_n, y_{n+1}),$$

which implies

$$e(y_n, y_{n+1}) \leq c_1 d(x_{n-1}, x_n)$$

if $e(y_n, y_{n+1}) \neq 0$. It follows that

$$d(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1}) \leq$$

$$\leq c_1 c_2 d(x_{n-1}, x_n) \leq \ldots \leq (c_1 c_2)^n\, d(x, x_1)$$

and since $0 \leq c_1 \cdot c_2 < 1$, $\{x_n\}$ is a Cauchy sequence with a limit z in X and $\{y_n\}$ is a Cauchy sequence with a limit w in Y.

Now, by using inequality (3), we have

$$e^2(Tz, y_n) \leq$$

$$c_1 \max\{d(z, x_{n-1})e(y_{n+1}, Tz), d(z, x_{n-1})e(y_{n-1}, y_n),$$

$$e(y_{n-1}, y_n)e(y_{n-1}, Tz)\}.$$

Letting n tend to infinity we have $e^2(Tz, w) \leq 0$ and so $Tz = w$. Similarly, we can prove that $Sw = z$ and so $STz = Sw = z, TSw = Tz = w$. Thus ST has a fixed point z and TS has a fixed point w. Now suppose that ST has a second fixed point z'. Then by using inequality (4), we have

$$d^2(z, z') = d^2(STz', STz) \leq$$

$$c_2 \max\{e(Tz', Tz)d(z, STz'), e(Tz', Tz), d(z, STz),$$

$$d(z, STz')d(z, STz)\} = c_2 e(Tz', Tz)d(z, z')$$

which implies $d(z',z) \leq c_2 e(Tz',Tz)$. But by using inequality (3),

$$e^2(Tz,Tz') = e^2(Tz',TSTz) \leq$$
$$c_1 \max\{d(z',STz)e(Tz,Tz'), d(z',STz)$$
$$e(Tz,TSTz), e(Tz,z)e(Tz,TSTz)\} = c_1 d(z',STz)e(Tz',Tz),$$

which implies

$$e(Tz',Tz) \leq c_1 d(z',STz) = c_1 d(z,z')$$

and so

$$d(z,z') \leq c_2 e(Tz',Tz) \leq c_1 c_2 d(z,z').$$

Since, $0 \leq c_1 c_2 < 1$, the uniqueness of z follows. Similarly, w is the unique fixed point of TS. If $\exists n \in N$ that $d(x_n, x_{n+1}) = 0$ or $e(y_n, y_{n+1}) = 0$ the theorem is evident.

Corollary 1. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself satisfying the inequalities*

$$d^2(Tx,TSy) \leq \tag{5}$$
$$c_1 \max\{d(x,Sy)d(y,Tx), d(x,Sy)d(y,TSy), d(y,Tx)d(y,TSy)\}$$

$$d^2(Sy,STx) \leq \tag{6}$$
$$c_2 \max\{d(y,Tx)d(x,Sy), d(y,Tx)d(y,STx), d(x,Sy)d(x,STx)\}$$

for all x, y in X, where $0 \leq c_1, c_2 < 1$, then ST has a unique fixed point z and TS has a unique fixed point w. Further $Tz = w$ and $Sw = z$ and if $z = w$, z is the unique fixed point of S and T.

Proof. The existence of z and w follows from Theorem 2. If $z = w$, then z is of course a common fixed point of S and T.

Now suppose that T has a second fixed point z'. Then, by using inequality (5), we have

$$d^2(z,z') = d^2(Tz',TSz) \leq$$
$$c_1 \max\{d(z',Sz)d(z,Tz'), d(z',Sz)d(z,TSz),$$
$$d(z,Tz')d(z,TSz)\} = c_1 d^2(z',z).$$

Since $0 \leq c_1 < 1$, the uniqueness of z follows. Similarly, z is the unique fixed point of S.

The proofs of Corollaries 2 and 3 follow easily.

Corollary 2. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$e^2(Tx,TSy) \leq$$
$$\leq a_1 d(x,Sy)e(y,Tx) + b_1 d(x,Sy)e(y,TSy) + c_1 e(y,Tx)e(y,TSy)$$
$$d^2(Sy,STx) \leq$$
$$\leq a_2 e(y,Tx)d(x,Sy) + b_2 d(x,STx)e(y,Tx) + c_2 d(x,Sy)d(x,STx)$$

for all x in X and y in Y, where $a_1,b_1,c_1,a_2,b_2,c_2 \geq 0$ and $(a_1+b_1+c_1)\cdot(a_2+b_2+c_2) < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Corollary 3. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself, satisfying the inequalities*

$$d^2(Tx,TSy) \leq$$
$$\leq a_1 d(x,Sy)d(y,Tx) + b_1 d(x,Sy)d(y,TSy) + c_1 d(y,Tx)d(y,TSy)$$
$$d^2(Sy,STx) \leq$$
$$\leq a_2 d(y,Tx)d(x,Sy) + b_2 d(y,Tx)d(x,STx) + c_2 d(x,Sy)d(x,STx)$$

for all x,y in X, where $a_1,b_1,c_1,a_2,b_2,c_2 \geq 0$ and $a_1+b_1+c_1 < 1$, $a_2+b_2+c_2 < 1$, then ST has a unique fixed point z and TS has a unique fixed point w. Further, $Tz = w$ and $Sw = z$ and if $z = w$, z is the unique fixed point of S and T.

Theorem 3. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$e(Tx,TSy)\cdot\max\{e(y,Tx),e(TSy,y)\} \tag{7}$$
$$\leq c_1 d(x,Sy)\cdot\max\{d(x,Sy),e(y,TSy)\}$$
$$d(Sy,STx)\cdot\max\{d(x,Sy),d(x,STx)\} \tag{8}$$
$$\leq c_2 e(y,Tx)\cdot\max\{e(y,Tx),d(x,STx)\}$$

for all x in X and y in Y, where $0 \leq c_1,c_2 < 1$ then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. Let x be an arbitrary point in X.Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by

$$x_n = (ST)^n x,\ y_n = T(ST)^{n-1}x$$

for $n = 1, 2 \ldots$ Using inequality (8), we have

$$d(x_n, x_{n+1}) \cdot \max\{d(x_n, x_{n+1}), d(x_n, x_n)\} \leq$$

$$\leq c_2 \cdot e(y_n, y_{n+1}) \cdot \max\{d(x_n, x_{n+1}), e(y_n, y_{n+1}\}.$$

and hence

$$d^2(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1}) d(x_n, x_{n+1}),$$

which implies

$$d(x_n, x_{n+1}) \leq c_2 e(y_n, y_{n+1})$$

if $d(x_n, x_{n+1}) \neq 0$ or $d^2(x_n, x_{n+1}) \leq c_2 e^2(y_n, y_{n+1})$, which implies

$$d(x_n, x_{n+1}) \leq \sqrt{c_2} \cdot e(y_n, y_{n+1}).$$

Since $c_2 \leq \sqrt{c_2}$, we have

$$d(x_n, x_{n+1}) \leq \sqrt{c_2} \cdot e(y_n, y_{n+1}).$$

Using inequality (7), we have

$$e(y_n, y_{n+1}) \cdot \max\{e(y_n, y_{n+1}), e(y_n, y_n)\} \leq$$

$$\leq c_1 d(x_{n-1}, x_n) \cdot \max\{d(x_{n-1}, x_n), e(y_n, y_{n+1})\}$$

and $e^2(y_n, y_{n+1}) \leq c_1 d(x_n, x_{n-1}) e(y_n, y_{n+1})$, which implies

$$e(y_n, y_{n+1}) \leq c_1 d(x_n, x_{n+1})$$

if $e(y_n, y_{n+1}) \neq 0$ or $e^2(y_n, y_{n+1}) \leq c_1 d^2(x_{n-1}, x_n)$, which implies

$$e(y_n, y_{n+1}) \leq \sqrt{c_1} \cdot d(x_{n-1}, x_n).$$

Since $c_1 \leq \sqrt{c_1}$, we have

$$e(y_n, y_{n+1}) \leq \sqrt{c_1} \cdot d(x_{n-1}, x_n).$$

It follows that

$$d(x_n, x_{n+1}) \leq \sqrt{c_2} \cdot e(y_n, y_{n+1}) \leq$$

$$\sqrt{c_1c_2}\, d(x_n, x_{n-1}) \leq \dots \leq$$
$$\leq (\sqrt{c_1c_2})^n d(x, x_1)$$

and since $0 \leq \sqrt{c_1c_2} < 1$, $\{x_n\}$ is a Cauchy sequence with a limit z in X and $\{y_n\}$ is a Cauchy sequence with a limit w in Y.

Now, by using inequality (7), we have

$$e(Tz, y_n) \cdot \max\{e(y_{n-1}, Tz), e(y_{n-1}, y_n)\} \leq$$

$$\leq c_1 \cdot d(z, x_{n-1}) \cdot \max\{d(z, x_{n-1}), e(y_n, y_{n+1})\}.$$

Letting n tend to infinity, we have $e^2(Tz, w) \leq 0$ and so $Tz = w$. Similarly, we can prove that $Sw = z$ and so

$$STz = Sw = z,\ TSw = Tz = w.$$

Thus ST has a fixed point z and TS has a fixed point w.

Now, suppose that ST has a second fixed point z'. Then by using inequality (8), we have

$$d(z, z') \cdot \max\{d(z, STz'), d(z, STz)\} = d(STz', STz)\cdot$$

$$\cdot \max\{d(z, STz'), d(z, STz)\} \leq c_2 e(Tz, Tz')\cdot$$
$$\cdot \max\{e(Tz, Tz'), d(z, STz)\} = c_2 e^2(Tz, Tz'),$$

which implies

$$d^2(z, z') \leq c_2 \cdot e^2(Tz, Tz').$$

But, by using inequality (7), we have

$$e(Tz, Tz') \cdot \max\{e(Tz, Tz'), e(Tz, TSTz)\} =$$

$$= e(Tz', TSTz) \cdot \max\{e(Tz, Tz'), e(TSTz, TSTz)\} \leq c_1 d(z', STz)\cdot$$
$$\cdot \max\{d(z', STz), e(Tz, TSTz)\} = c_1 d^2(z', STz) = c_1 d^2(z, z'),$$

which implies

$$e^2(Tz', Tz) \leq c_1 \cdot d^2(z, z').$$

It follows that

$$d^2(z, z') \leq c_2 e^2(Tz, Tz') \leq c_1 c_2 d^2(z, z')$$

and so

$$d^2(z,z') \leq c_1c_2d^2(z,z').$$

Since $0 \leq c_1c_2 < 1$, the uniqueness of z follows. Similarly, w is the unique fixed point of TS.

If $\exists n \in N$ that $d(x_n, x_{n+1}) = 0$ or $e(y_n, y_{n+1}) =$ the theorem is evident.

Corollary 4. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself satisfying the inequalities*

$$d(Tx,TSy)\max\{d(y,Tx),d(y,TSy)\} \leq \tag{9}$$

$$c_1d(x,Sy)\max\{d(x,Sy),d(y,TSy)\}$$

$$d(Sy,STx)\max\{d(x,Sy),d(x,STx)\} \leq \tag{10}$$

$$c_2d(y,Tx)\max\{d(y,Tx),d(x,STx)\}$$

for all x,y in X, where $0 \leq c_1,c_2 < 1$, then ST has a unique fixed point z and TS has a unique fixed point w. Further, $Tz = w$ and $Sw = z$ and if $z = w$, z is the unique fixed point of S and T.

Proof. The existence of z and w follows from Theorem 3. If $z = w$, the z is of course a common fixed point of S and T.

Now, suppose that T has a second fixed point z'. Then, by using inequality (9), we have

$$d(z,z')\cdot\max\{d(z',Sz),d(z,TSz)\} =$$

$$= d(Tz',TSz)\cdot\max\{d(z',Sz),d(z,TSz)\} \leq$$

$$\leq c_1d(z,Tz')\cdot\max\{d(z,Tz'),d(z,TSz)\} = c_1d^2(z,z'),$$

which implies

$$d^2(z,z') \leq c_1d^2(z,z').$$

Since, $0 \leq c_1 < 1$, the uniqueness of z follows. Similarly, z is the unique fixed point of S.

The proofs of Corollaries 5 and 6 follow easily.

Corollary 5. *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X, satisfying the inequalities*

$$e(Tx,TSy)\cdot\max\{e(y,Tx),e(y,TSy)\}$$

$$\leq d(x,Sy)[a_1d(x,Sy)+b_1e(y,TSy)]$$

$$d(Sy,STx)\cdot\max\{d(x,Sy),d(x,STx)\}\leq e(y,Tx)$$

$$[a_2e(y,Tx)+b_2d(x,STx)]$$

for all x in X and y in Y, where $0\leq a_i+b_i<1,\ i=1,2,$ then TS has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz=w$ and $Sw=z$.

Corollary 6. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself, satisfaying the inequalities*

$$d(Tx,TSy)\cdot\max\{d(y,Tx),d(y,TSy)\}\leq$$

$$d(x,Sy)[a_1d(x,Sy)+b_1d(y,TSy)]$$

$$d(Sy,STx)\cdot\max\{d(x,Sy),d(x,STx)\}\leq$$

$$d(y,Tx)[a_2d(y,Tx)+b_2d(x,STx)]$$

for all x,y in X, where $0\leq a_i+b_i<1,\ i=1,2,$ then ST has a unique fixed point z and TS has a unique fixed point w. Further, $Tz=w$ and $Sw=z$ and if $z=w$, z is the unique fixed point of S and T.

Remark. A condition of the following type was used in [4]:

$$d(Tx,Sy)\leq f(d(x,y),d(x,Sx),d(y,Ty),d(x,Sy),d(y,Tx))$$

with a semi-homogeneous function f, $f:R_+^5\to R_+$, satisfying some additional assumption.Similarly, conditions (3) and (4) could be written down in the same form with

$$f(s,t,r,m,p,q)=\max\{st,sm,pq\}/q,\ q\neq 0.$$

A similar connection is between [5] and conditions(7) and (8).

References

[1] Fisher B.: Fixed point on two metric spaces, Glasnik matem. 16 (36) (1981), 333-337

[2] Fisher B.: Related fixed points on two metric spaces, Mathematics Seminar Notes, 10 (1982), 17-26.

[3] Hadžić O.: Fixed point theorems for multivalued mapping in metric spaces, Anal. Univ. Bucureşti (1984), 30-37.

[4] Tasković M.: On a family of contractive maps, Bull.Austral. Math. Soc. 13 (1975), 301-308.

[5] Tasković M.: On the convergence of certain sequences and some applications - II, Publ. Inst. Math. 35 (1977), 271-281.

REZIME

NEPOKRETNE TAČKE NA DVA KOMPLETNA METRIČKA PROSTORA

Neka su (X,d) i (Y,e) dva kompletna metrička prostora. Dokazano je da ako je T preslikavanje X u Y i ako je S preslikavanje Y u X tako da su zadovoljene nejednakosti

$$e^2(Tx,TSy) \leq$$

$$c_1 \max\{d(x,Sy)e(y,Tx), d(x,Sy)e(y,TSy), e(y,Tx)e(y,TSy)\}.$$

$$d^2(Sy,STx) \leq$$

$$c_2 \max\{e(y,Tx)d(x,Sy), e(y,Tx)d(x,STx), d(x,Sy)d(x,STx)\}.$$

za sve x u X i y u Y, gde je $0 \leq c_1 \cdot c_2 < 1$ ili nejednakosti

$$e(Tx,TSy) \cdot \max\{e(y,Tx), e(TSy,y)\}$$

$$\leq c_1 d(x,Sy) \cdot \max\{d(x,Sy), e(y,TSy)\}$$

$$d(Sy,STx) \cdot \max\{d(x,Sy), d(x,STx)\} \leq$$

$$\leq c_2 e(y,Tx) \cdot \max\{e(y,Tx), d(x,STx)\}$$

za sve x u X i y u Y , gde je $0 \leq c_1, c_2 < 1$, onda ST ima jedinstvenu nepokretnu tačku z u X i TS ima jedinstvenu nepokretnu tačku w u Y. Dalje, $Tz = w$ i $Sw = z$.

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