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THE SEQUENCE SPACE $Ces(p, s)$ AND RELATED MATRIX TRANSFORMATIONS

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Abstract

In the paper, the main purpose is to define and to investigate the sequence space *Ces(p,s)* and to determine the matrices of classes $(Ces(p,s), \ell_\infty)$ and $(Ces(p,s), c)$ where ℓ_∞ and c are respectively the space of bounded and convergent complex sequences and for $p = (p_r)$ with inf $p_r > 0$, the space $Ces(p,s)$ is defined for $s \geq 0$ by

$$Ces(p,s) = \{x = (x_k) : \sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_r |x_k|)^{p_r} < \infty\},$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq k < 2^{r+1}$.

These spaces i.e. $Ces(p,s)$ can be viewed as $Ces(p)$ spaces with weights, generalizing $Ces(p)$ spaces.

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1.

Let $A = (a_{n,k})$ be an infinite matrix of complex numbers $a_{n,k}(n, k = 1,2,\ldots)$ and V, W be two subsets of the space of complex sequences. We say that the matrix A defines a matrix transformations from V into W and denote it by $A \in (V, W)$, if for every sequence $x = (x_k) \in V$ the sequence $A(x) = A_n(x)$ is in W, where $A_n = \sum_{k=1}^{\infty} a_{n,k} x_k$.

The main purpose of this note is to define and to investigate the sequence space $Ces(p,s)$ and to determine the matrices of classes $(Ces(p,s), \ell_\infty)$ and $(Ces(p,s), c)$, where ℓ_∞ and c are respectively the space of bounded and convergent complex sequences and for $p = (p_r)$ with inf $p_r > 0$, the space $Ces(p,s)$ is defined for $s \geq 0$ by

$$Ces(p,s) = \{x = (x_k) : \sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_r |x_k|)^{p_r} < \infty\},$$

where $\sum_r$ denotes a sum over the ranges $2^r \leq k < 2^{r+1}$.

Obviously, the sequence space

$$Ces(p) = \{x = (x_k) : \sum_{r=0}^{\infty} (\frac{1}{2^r} \sum_r |x_k|)^{p_r} < \infty\}, \text{ where inf } p_r > 0,$$

which has been investigated by K.P. Lim ([3], [4]) is a special case of $Ces(p,s)$ which corresponds to $s = 0$. And $Ces(p,s) \supset Ces(p) \supset \ell(p)$ for $p_r \geq 1$.

With regard to notation, the dual space of $Ces(p,s)$, i.e. the space of all the continuous linear functionals on $Ces(p,s)$, will be denoted by $Ces^*(p,s)$. We write $A_r(n) = \max_r |a_{n,k}|$ where for each n the maximum is taken for k in $[2^r, 2^{r+1}]$.

Troughout the paper the following well-known inequality (See[1]) will be used frequelntly.

For any $C > 0$ and any two complex numbers a, b:

(1) $$|ab| \leq C(|a|^q C^{-q} + |b|^p) \text{ where } p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

To begin with we can show that the space $Ces(p,s)$ is paranormed by

(2) $$g(x) = (\sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_r |x_k|)^{p_r})^{1/M}$$

if $H = \sup_r p_r < \infty$ and $M = \max(1, H)$. Clearly $g(\Theta) = 0$ and $g(x) = g(-x)$, where $\Theta = (0, 0, \ldots\ldots)$. Take any $x, y \in Ces(p,s)$. Since

$p_r \leq M$ and $M \geq 1$, using Minkowski's inequality we obtain that g is subadditive. Finally, we have to check the continuity of scalar multiplication. For any complex λ, we have

$$g(\lambda x) = [\sum_{r=0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}|\lambda x_k|)^{p_r}]^{1/M}$$

$$\leq \sup_r |\lambda|^{p_r/M} \cdot g(x).$$

Now let $\lambda \to 0$ for any fixed x with $g(x) \neq 0$. Since $\sum_{r=0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |x_k|)^{p_r} < \infty$, there exists an integer $m_0 > 1$, for $|\lambda| < 1$ and $\epsilon > 0$, such that

$$\sum_{r=m_0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |\lambda x_k|)^{p_r} < \epsilon. \tag{3}$$

Taking $|\lambda|$ sufficiently small such that $|\lambda|^{p_r} < \epsilon/g(x)$ for $r = 0, 1, \ldots\ldots m_0 - 1$, we then have

$$\sum_{r=0}^{m_0-1}(2^r)^{-s}(\frac{1}{2^r}\sum_r |\lambda x_k|)^{p_r} < \epsilon. \tag{4}$$

(3) and (4) together implies that $g(\lambda x) \to 0$ as $\lambda \to 0$.

It is quite routine to show that $(Ces(p,s), d)$ is a metric space with the metric $d(x,y) = g(x-y)$ provided that $x, y \in Ces(p,s)$, where g is defined by (2). And using a similar method to that in [2] one can show that $Ces(p,s)$ is complete under the metric mentioned above.

2.

Now we are going to give the following theorem by which the Köthe-Toeplitz dual of $Ces(p,s)$ will be determined.

Theorem 1. *If* $1 < p_r \leq \sup_r p_r < \infty$ *and* $\frac{1}{p_r} + \frac{1}{q_r} = 1$, $r = 0, 1, 2, \ldots$ *then*

$$Ces^+(p,s) = \{a = (a_k) : \sum_{r=0}^{\infty}(2^r)^{s(q_r-1)}(2^r \max_r |a_k|)^{q_r} E^{-q_r} < \infty$$

$$\textit{for some integer } E > 1\}, \quad s \geq 0.$$

Proof. Let $1 < p_r \le \sup_r p_r < \infty$ and $\frac{1}{p_r} + \frac{1}{q_r} = 1$, for $r = 0, 1, 2, \ldots$. Then take

$$\mu(q,s) = \{a = (a_k) : \sum_{r=0}^{\infty} (2^r)^{s(q_r - 1)} (2^r \max_r |a_k|)^{q_r} E^{-q_r} < \infty \tag{5}$$

$$\text{for some integer } E > 1\}, \quad s \ge 0.$$

We now want to show that $Ces^+(p,s) = \mu(q,s)$. Let $x \in Ces(p,s)$, $a \in \mu(q,s)$. Therefore using inequality (1), we get

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_k x_k| &= \sum_{r=0}^{\infty} \sum_{r} |a_k x_k| \\
&\le \sum_{r=0}^{\infty} 2^r \max_r |a_k| (2^r)^{s/p_r} \cdot \frac{1}{2^r} (2^r)^{-s/p_r} \sum_r |x_k|
\end{aligned}$$

$$\le E\Big(\sum_{r=0}^{\infty} (2^r \max_r |a_k|)^{q_r} (2^r)^{sq_r/p_r} E^{-q_r} + \sum_{r=0}^{\infty} (2^r)^{-s} \big(\frac{1}{2^r} \sum_r |x_k|\big)^{p_r}\Big).$$

So $\sum |a_k x_k|$ is convergent, which implies that $\sum a_k x_k$ is convergent i.e. $a \in Ces^+(p,s)$. In other words $Ces^+(p,s) \supset \mu(q,s)$. Conversely, let us suppose that $\sum a_k x_k$ is convergent and $x \in Ces(p,s)$, but $a \notin \mu(q,s)$. Then we write that

$$\sum_{r=0}^{\infty} (2^r)^{s(q_r-1)} (2^r \max_r |a_k|)^{q_r} E^{-q_r} = \infty \text{ for each } s \ge 0$$

and for every $E > 1$. So we can find a sequence $0 = n(0) < n(1) < n(2) < \ldots$ such that for $\nu = 0, 1, 2, \ldots$

$$M_\nu = \sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r)^{s(q_r-1)} (2^r \max_r |a_k|)^{q_r} (\nu+2)^{-q_r/p_r} > 1.$$

Now define a sequence $x = (x_k)$ as follows: for each ν,

$$x_{N(r)} = 2^{rq_r} |a_{N(r)}|^{q_r - 1} \, sgn \, a_{N(r)} (2^r)^{s(q_r-1)} (\nu+2)^{-q_r} M_\nu^{-1},$$

for $n(\nu) \le r \le n(\nu+1) - 1$, and $x_k = 0$ for $k \ne N(r)$ where $N(r)$ is such that $|a_{N(r)}| = \max_r |a_k|$, the maximum is taken for k in $[2^r, 2^{r+1})$.

Therefore,

$$\sum_{k=2^{n(\nu)}}^{2^{n(\nu+1)-1}} a_k x_k = \sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r|a_{N(r)}|)^{q_r}(2^r)^{s(q_r-1)}(\nu+2)^{-q_r}M_\nu^{-1} =$$

$$= M_\nu^{-1}(\nu+2)^{-1} \sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r|a_{N(r)}|)^{q_r}\cdot(2^r)^{s(q_r-1)}(\nu+2)^{-q_r/p_r} =$$

$$= (\nu+2)^{-1}.$$

It follows that $\sum_{k=1}^{\infty} a_k x_k = \sum_{\nu=0}^{\infty}(\nu+2)^{-1}$ diverges. Moreover

$$\sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r)^{-s}\cdot(\frac{1}{2^r}\sum_r |x_k|)^{p_r} =$$

$$= \sum_{r=n(\nu)}^{n(\nu+1)-1} (2^r)^{-s}(2^{r(q_r-1)p_r}|a_{N(r)}|^{(q_r-1)p_r}(\nu+2)^{-q_r p_r}.$$

$$\cdot M_\nu^{-p_r}(2^r)^{s(q_r-1)p_r})$$

$$\leq (\nu+2)^{-2}M^{-1} \sum_{r=n(\nu)}^{n(\nu+1-1)} (2^r)^{s(q_r-1)}2^{rq_r}|a_{N(r)}|^{q_r}(\nu+2)^{-q_r/p_r}$$

$$= (\nu+2)^{-2}$$

Hence, $\sum_{r=0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |x_k|)^{p_r} \leq \sum_{\nu=0}^{\infty}(\nu+2)^{-2} < \infty$ i.e. $x \in Ces(p,s)$. And this contradicts our assumption. So $a \in \mu(q,s)$ i.e. $\mu(q,s) \supset Ces^+(p,s)$. Then combining these two results we get

$$Ces^+(p,s) = \mu(q,s).$$

Let us now determine the continuous dual of $Ces(p,s)$ by the following theorem.

Theorem 2. *Let* $1 < p_r \leq \sup_r p_r < \infty$. *Then* $Ces^*(p,s)$ *is isomorphic to* $\mu(q,s)$ *which is defined by (5).*

Proof. It is easy to check that each $x \in Ces(p,s)$ can be written as $x = \sum_{k=1}^{\infty} x_k e_k$ where $e_k = (0,0,\dots,1,0,0\dots)$, where 1 appears at k-th place.

Then for any $f \in Ces^*(p,s)$

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k a_k,$$

where $f(e_k) = a_k$.

By theorem 1, the convergence of $\sum_{k=1}^{\infty} a_k x_k$ for every x in $Ces(p,s)$ implies that $a \in \mu(q,s)$.

If $x \in Ces(p,s)$ and if we take $a \in \mu(q,s)$ taken by Theorem 1, $\sum_{k=1}^{\infty} a_k x_k$ converges and clearly defines a linear functional on $Ces(p,s)$. Using the same kind of argument to that in Theorem 1, it is easy to check that

$$\sum_{k=1}^{\infty} |a_k x_k| \leq E(\sum_{r=0}^{\infty}(2^r)^{s(q_r-1)}(2^r \max_r |a_k|)^{q_r} E^{-q_r} + 1)g(x)$$

whenever $g(x) \leq 1$. Hence $\sum_{k=1}^{\infty} a_k x_k$ defines an element of $Ces^*(p,s)$. Obviously, the map $T : Ces^*(p,s) \to \mu(q,s)$ given by $T(f) = (a_1, a_2, \dots)$ is linear and bijective. Hence $Ces^*(p,s)$ is isomorphic to $\mu(q,s)$.

3.

In the following theorems we are going to characterize the matrix classes $(Ces(p,s), \ell_\infty)$ and $(Ces(p,s), c)$.

Theorem 3. *Let* $1 < p_r \leq \sup_r p_r < \infty$. *Then* $A \in (Ces(p,s), \ell_\infty)$ *iff there exists an integer* $E > 1$ *such that* $U(E) > \infty$ *where*

$$U(E) = \sup_n \sum_{r=0}^{\infty} (2^r A_r(n))^{q_r} (2^r)^{s(q_r-1)} E^{-q_r} < \infty$$

and

$$\frac{1}{p_r} + \frac{1}{q_r} = 1, \quad r = 0,1,2,\dots.$$

Proof. Sufficiency. Suppose there exists an integer $E > 1$ and consider the hypothesis. Then by inequality (1), we have

$$\sum_{k=1}^{\infty} |a_{n,k} x_k| = \sum_{r=0}^{\infty} \sum_{r} |a_{n,k} x_k|$$

$$\leq E(\sum_{r=0}^{\infty} (2^r A_r(n))^{q_r} (2^r)^{sq_r/p_r}$$

$$E^{-q_r} + \sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_{r} {}^{n} x_k|)^{p_r}) < \infty.$$

Therefore $A \in (Ces(p,s), \ell_\infty)$.

Necessity. Suppose that $A \in (Ces(p,s), \ell_\infty)$ but that

$$\sup_n \sum_{r=0}^{\infty} (2^r A_r(n))^{q_r} (2^r)^{s(q_r - 1)} E^{-q_r} = \infty \text{ for every integer } E > 1.$$

Then $\sum_{k=1}^{\infty} a_{n,k} x_k$ convergences for every n and for every $x \in Ces(p,s)$, whence $(a_{n,k})_{k=1,\dots} \in Ces^{+}(p,s)$ for every n.

By Theorem 1, it follows that each A_n defined by $A_n(x) = \sum_{k=1}^{\infty} a_{n,k} x_k$ is an element of $Ces^*(p,s)$. Since $Ces(p,s)$ is complete and since $\sup_n |A_n(x)| < \infty$ on $Ces(p,s)$, by the uniform boundedness principle there exists a number L independent of n and x and a number $\delta < 1$ such that

(6) $$|A_n(x)| \leq L$$

for every $x \in S[\Theta, \delta]$ and every n, where $S[\Theta, \delta]$ is the closed sphere in $Ces(p,s)$ with centre the origin Θ and radius δ.

Now choose an integer $Q > 1$ such that

$$Q\delta^M > L.$$

Since

$$\sup_n \sum_{r=0}^{\infty} (2^r A_r(n))^{q_r} (2^r)^{s(q_r - 1)} Q^{-q_r} = \infty,$$

there exists an integer $m_0 > 1$ such that

(7) $$R = \sum_{r=0}^{m_0} (2^r A_r(n))^{q_r} (2^r)^{s(q_r - 1)} Q^{-q_r} > 1.$$

Define a sequence $x = (x_k)$ as follows:

$$x_k = 0 \text{ if } k \geq 2^{m_0+1}$$

and

$$x_{N(r)} = 2^{rq_r}\delta^{M/p_r}(sgn\ a_{n,N(r)})$$
$$|a_{n,N(r)}|^{q_r-1}R^{-1}Q^{-q_r/p_r}(2^r)^{s(q_r-1)}$$

$x_k = 0$ ($k \neq N(r)$, for $0 \leq r \leq m_0$) where $N(r)$ is the smallest integer such that $|a_{n,N(r)}| = \max_r |a_{n,k}|$. Then one can easly show that $g(x) \leq \delta$ but $|A_n(x)|^r > L$, which contradicts to (6).

This completes the proof of the Theorem.

Theorem 4. *Let* $1 < p_r \leq \sup_r p_r < \infty$. *Then* $A \in (Ces(p,s),c)$ *if*

(4.1) $a_{n.k} \to \alpha_k (n \to \infty, k$ *fixed*)

(4.2) *there exists an integer* $E > 1$ *such that* $U(E) < \infty$, *where* $U(E)$*is defined as in Theorem 3.*

Proof. Suppose $A \in (Ces(p,s),c)$. Then $A_n(x)$ exists for each $n \leq 1$ and $\lim_n A_n(x)$ exists, for every $x \in Ces(p,s)$. Therefore, by a similar argument to that in Theorem 3, we have condition (4.2). The condition (4.1) is obtained by taking $x = e_k \in Ces(p,s)$, where $e_k = (0,0,\ldots 1,0,0,0,\ldots)$ where 1 appearsat k-th place.

For the sufficiency the conditions of the Theorem imply that

$$\sum_{r=0}^{\infty}(2^r \max_r |\alpha_k)^{q_r}(2^r)^{s(q_r-1)} \tag{8}$$

$$E^{-q_r} \leq U(E) < \infty.$$

By using (8), it is easy to check that $\sum_{k=1}^{\infty} \alpha_k x_k$ is absolutely convergent for each $x \in Ces(p,s)$. Moreover, for each $x \in Ces(p,s)$, there exists an integer $m_0 \geq 1$, such that

$$g_{m_0}(x) = \sum_{r=m_0}^{\infty}(2^r)^{-s}(\frac{1}{2^r}\sum_r |x_k|)^{p_r} < 1.$$

If $g_{m_0}(x) \neq 0$ the by the proof of Theorem 2 and by inequality (1) we have

$$\sum_{\substack{m_0 \\ k=2}}^{\infty} |a_{n,k} - \alpha_k|\,|x_k| \leq E\Big(\sum_{r=m_0}^{\infty} (2^r)^{s(q_r-1)} (2^r B_r(n))^{q_r} E^{-q_r} + 1\Big) g_{m_0}(x)^{1/M}, \tag{9}$$

where $B_r(n) = \max_r |a_{n,k} - \alpha_k|$.

Cleraly (9) holds if $g_{m_0}(x) = 0$. Since

$$\sum_{r=m_0}^{\infty} (2^r)^{s(q_r-1)} (2^r B_r(n))^{q_r} E^{-q_r} \leq 2\,U(E) < \infty,$$

from (9), it follows immediately that $\lim_{n\to\infty} \sum a_{n,k} x_k = \sum \alpha_k x_k$. This shows that $A \in (Ces(p,s), c)$ which proves the Theorem.

Remark. To be able to get the necessary and sufficient confition for $A \in (Ces(p,s), c_0)$, where c_0 is the space of null sequences, it would be enough to take $\alpha_k = 0$ in the above theorem.

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REZIME

NIZOVNI PROSTOR $Ces(p, s)$ I ODNOSNE MATRIČNE TRANSFORMACIJE

Glavna svrha rada je da se definiše i ispita nizovni prostor $Ces(p, s)$, te da se odrede matrice klasa $(Ces(p, s), \ell_\infty)$ i $(Ces(p, s), c)$ gde su ℓ_∞ i c prostori ograničenih, odnosno konvergentnih kompleksnih nizova, a za $p = (p_2)$ sa $\inf p_2 > 0$, prostor $Ces(p, s)$ je definisan za $s \geq 0$ sa:

$$Ces(p, s) = \{x = (x_k) : \sum_{r=0}^{\infty} (2^r)^{-s} (\frac{1}{2^r} \sum_r |x_k|)^{p_r} < \infty\},$$

gde $\sum_r$ označava sumu po svim k:

$$z^r \leq k \leq 2^{r+1}.$$

Prostori $Ces(p, s)$ se mogu posmatrati kao $Ces(p)$ prostori sa težinama, koji uopštavaju $Ces(p)$ prostore.

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