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UNIFORM CONVEXITY AND THE FIXED POINT PROPERTY[1]

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Abstract

In this paper the concept of (F^k) uniformly convex spaces and some fixed point theorems for generalized nonexpansive mappings are introduced and presented. The results obtained in this paper extend and improve some recent results of Dulst [3], Jaggi [5] and Browder [2], Kirk [6].

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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. A mapping $T : K \to K$, $K \subset X$ is said to be nonexpansive, if

$$(1.1) \qquad \|Tx - Ty\| \le \|x - y\|, \qquad \forall x, y \in K.$$

$T : K \to K$ is said to be generalized expansive, if for every subset E of K with at least two points and $T(E) \subset E$

$$(1.2) \qquad \sup_{y \in E} \|Tx - Ty\| \le \sup_{y \in E} \|x - y\|, \qquad \forall x \in E.$$

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It is easy to see that the nonexpansive mapping and the mapping $T : K \to K$ satisfying one of the following conditions:

$$(1.3) \qquad \|Tx - Ty\| \le \sup\{\|x - z\|,\ z \in \{T^n y\}_{n \ge 0}\}, \qquad x, y \in K$$

$$(1.4) \quad \|Tx - Ty\| \le \sup\{\|x - z\|,\ z \in (\{T^n x\}_{n \ge 0} \cup \{T^n y\}_{n \ge 0})\},\ x, y \in K$$

are all the special cases of generalized nonexpansive mappings.

Let $(X, \|\cdot\|)$ be a Banach space, and $K \subset X$ a nonempty bounded closed convex subset. K is said to have normal structure, if for each closed convex subset $C \subset K$ consisting of more than one point there exists a point $x \in C$ such that

$$\sup\{\|x - y\|,\ y \in C\} < \operatorname{diam}(C) \qquad \text{(the diameter of } C).$$

A Banach space X is said to have normal structure, if for each bounded closed nonempty convex set $K \subset X$ it has the normal structure.

A Banach space X is said to have the fixed point property (F.P.P.) for nonexpansive mappings (or for generalized nonexpansive mappings), if for every weakly compact convex subset K of X and for every nonexpansive mapping (or generalized nonexpansive mapping) $T : K \to K$, T has a fixed point in K.

The following two questions have long been open.

1. Which Banach spaces have the F.P.P. for generalized nonexpansive mappings?

2. Which Banach spaces have the F.P.P. for nonexpansive mapping?

It is obvious that question 2 is a special case of question 1.

Concerning question 2, recently Alspach [1] showed that the answer is negative for $X = L^1[0,1]$. If X is a Hilbert, or more general, if X is a reflexive Banach space having normal structure then the answer to question 2 is also positive (cf. Browder [2] and Kirk [6]).

Concerning question 1, recently Jaggi [5] showed that, if X is a reflexive Banach space having normal structure then the answer is also affirmative.

The purpose of this paper is first to introduce the concept of (F^k) uniformly convex space and then use this concept to prove that every Banach space with the separated property (S.P.) can be reendowed with an equivalent norm $\|\cdot\|_1$ so that $(X, \|\cdot\|_1)$ has the F.P.P. for generalized nonexpansive mapping (hence for every kind of mappings in (1.1), (1.3) or (1.4)). The results presented in this paper extend and improve some recent results of [2,3,5,6].

2. Definition and preliminaries

Let X be a Banach space, and X^* its dual. Throughout this paper we always denote by

$$B(X) = \{x \in X \;:\; \|x\| \le 1\}\,;$$

$$B(X^*) = \{f \in X^* \;:\; \|f\| \le 1\}\,;$$

$$S(X^*) = \{f \in X^* \;:\; \|f\| = 1\}\,.$$

Definition 1. *A Banach space $(X, \|\cdot\|)$ is said to have the separated property (S.P.), if there exists a countable set $F \subset B(X^*)$ such that if an $x \in X$ satisfies $f(x) = 0$ for every $f \in F$, then $x = 0$.*

In what folows we denote such kind of spaces by $(X, \|\cdot\|, F)$.

Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. . By virtue of F we can define $\|\cdot\|_1$ as follows:

$$\|x\|_1 = \left(\|x\|^2 + \sum_{i=1}^{\infty} \frac{|f_i(x)|^2}{2^i}\right)^{\frac{1}{2}}, \qquad f_i \in F. \tag{2.1}$$

It is easy to prove that $\|\cdot\|_1$ is a norm on X equivalent to the given norm $\|\cdot\|$, and

$$\|x\| \le \|x\|_1 \le \sqrt{2}\|x\|. \tag{2.2}$$

Example. Let $(X, \|\cdot\|)$ be a separable Banach space. Then $(X, \|\cdot\|)$ is an example of Banach space with the S.P. . In fact, let $\{x_n\}$ be a countable dense subset of X. By Hahn-Banach Theorem, for each x_n, $n = 1, 2, \dots$ there exists an $f_n \in S(X^*) \subset B(X^*)$ such that $f_n(x_n) = \|x_n\|$. Letting

$$F = \{f_n\}_{n=1}^{\infty} \tag{2.3}$$

we know that if $x \in X$ satisfies $f(x) = 0$, for every $f \in F$, then $x = 0$. In fact, let $\{x_{n_i}\} \subset \{x_n\}$ and $x_{n_i} \to x$ hence we have

$$\|x_{n_i}\| = f_{n_i}(x_{n_i}) - f_{n_i}(x) \le \|f_{n_i}\| \cdot \|x_{n_i} - x\| \to 0 \ \ (n_i \to \infty).$$

This implies that $x_{n_i} \to 0$, i.e. $x = 0$.

Definition 2. *Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. . X is said to be (F^k) uniformly convex for some positive integer k, if for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $k+1$ points $x_1, x_2, \ldots, x_{k+1} \in B(X)$, if*

$$\|x_1 + x_2 + \cdots + x_{k+1}\| > (k+1) - \delta,$$

then

$$|A(x_1, x_2, \ldots, x_{k+1}; f_1, \ldots, f_k)| < \varepsilon, \qquad \forall \{f_1, f_2, \ldots, f_k\} \subset F,$$

where

$$A(x_1, x_2, \ldots, x_{k+1}; f_1, \ldots, f_k) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_{k+1}) \end{vmatrix}.$$

Proposition 1. *If $(X, \|\cdot\|, F)$ is (F^k) uniformly convex for some positive integer k, then $(X, \|\cdot\|, F)$ is (F^{k+1}) uniformly convex.*

Proof. Since X is (F^k) uniformly convex, hence for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $x_1, x_2, \ldots, x_{k+1}$ are any $k+1$ points of $B(X)$ with $\|x_1 + x_2 + \cdots + x_{k+1}\| > (k+1) - \delta$, then

(2.4) $$|A(x_1, x_2, \ldots, x_{k+1}; f_1, \ldots, f_k)| < \frac{\varepsilon}{k+1}, \qquad \forall \{f_1, \ldots, f_k\} \subset F.$$

Now suppose that $z_1, z_2, \ldots, z_{k+2}$ are any $k+2$ points in $B(X)$ with

$$\|z_1 + z_2 + \cdots + z_{k+2}\| > (k+2) - \delta.$$

Then from the triangle inequality for each i we have

(2.5) $$\|z_1 + z_2 + \cdots + z_{i-1} + z_{i+1} + \cdots + z_{k+2}\| \ge$$

$$\geq \|z_1 + z_2 + \cdots + z_{k+2}\| - \|z_i\| > (k+1) - \delta.$$

Now let $f_1, f_2, \ldots, f_{k+1}$ be any $k+1$ functionals of F and consider the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(z_1) & f_1(z_2) & \cdots & f_1(z_{k+2}) \\ \vdots & \vdots & & \vdots \\ f_{k+1}(z_1) & f_{k+1}(z_2) & \cdots & f_{k+1}(z_{k+2}) \end{vmatrix}$$

Expanding in minors along the second row and using (2.4) and (2.5) we have

$$|A(z_1, z_2, \ldots, z_{k+2}; f_1, \ldots, f_{k+1})| < \varepsilon, \qquad \forall \{f_1, \ldots, f_{k+1}\} \subset F.$$

This completes the proof. □

Proposition 2. *Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. and C a subset of X. Suppose that there exists some k such that for any $k+1$ points $x_1, x_2, \ldots, x_{k+1} \in C$ the following holds:*

$$A(x_1, x_2, \ldots, x_{k+1}; f_1, \ldots, f_k) = 0, \qquad \forall \{f_1, \ldots, f_k\} \subset F.$$

Then the dimension of span(C) dim(span(C)) $< \infty$.

The proof of this proposition is straightforward we omit the statement here.

Proposition 3. [7] *If E is the real or complex field, then for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) \in (0,1)$ such that for any $|x| \leq 1$, $|y| \leq 1$, and $|x - y| \geq \varepsilon$, the following holds*

$$\left|\frac{x+y}{2}\right|^2 \leq (1-\delta)\frac{|x|^2 + |y|^2}{2}.$$

Theorem 1. *If $(X, \|\cdot\|, F)$ is a (F^k) uniformly convex Banach space for some positive integer k, then X has normal structure.*

Proof. Let $K \subset X$ be any nonempty bounded closed convex subset with $\operatorname{diam}(K) = d > 0$. If the dimension of K is finite, then K is a compact convex subset of X. By a wellknown result we know that K has normal structure. Therefore without loss of generality, we can suppose that

$\dim(K) = \infty$. By virtue of Proposition 2 there exist some $x_1^*, x_2^*, \ldots, x_{k+1}^* \in K$, $f_1^*, f_2^*, \ldots, f_k^* \in F$ and $\varepsilon_0 > 0$ such that

(2.6) $$\frac{1}{d^k}|A(x_1^*, x_2^*, \ldots, x_{k+1}^*; f_1^*, \ldots, f_k^*)| \geq \varepsilon_0.$$

Since X is (F^k) uniformly convex, therefore for ε_0 there exists a $\delta(\varepsilon_0) > 0$ such that for any $y_1, y_2, \ldots, y_{k+1} \in B(X)$ if

(2.7) $$|A(y_1, y_2, \ldots, y_{k+1}; f_1^*, \ldots, f_k^*)| \geq \varepsilon_0,$$

then

(2.8) $$\|y_1 + \cdots + y_{k+1}\| \leq (k+1) - \delta(\varepsilon_0).$$

Next for any $z \in K$ it is easy to see that $\frac{1}{d}(x_i^* - z) \in B(X)$, $i = 1, 2, \ldots, k+1$ and from (2.6) we have

(2.9) $$|A(\frac{x_1^* - z}{d}, \ldots, \frac{x_{k+1}^* - z}{d}; f_1^*, \ldots, f_k^*)| =$$

$$= \frac{1}{d^k}|A(x_1^*, \ldots, x_{k+1}^*; f_1^*, \ldots, f_k^*)| \geq \varepsilon_0.$$

In view of (2.7) and (2.8) we have

$$\left\|\frac{x_1^* - z}{d} + \cdots + \frac{x_{k+1}^* - z}{d}\right\| \leq (k+1) - \delta(\varepsilon_0),$$

i.e. $$\left\|z - \frac{x_1^* + x_2^* + \cdots + x_{k+1}^*}{k+1}\right\| \leq d - \frac{d}{k+1}\delta(\varepsilon_0) < \operatorname{diam}(K).$$

This implies that $X, \|\cdot\|)$ has normal structure.

This completes the proof. □

Theorem 2. *Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. . If $\|\cdot\|_1$ is the equivalent norm on X defined by (2.1) then $(X, \|\cdot\|_1, F)$ is a (F^k) uniformly convex space for any positive integer k.*

Proof. Firstly, we show that $(X, \|\cdot\|_1, F)$ has the S.P. . For the purpose, it suffices to prove

$$F \subset B_1(X^*) \triangleq \{f \in X^* : \|f\|_1 \leq 1\},$$

where $B_1(X) \triangleq \{x \in X : \|x\|_1 \leq 1\}$, and

$$\|f\|_1 = \sup_{x \in B_1(X)} \{|f(x)|\}.$$

In fact, if $f \in F$, it follows from (2.2) that

$$\|f\|_1 = \sup_{\|x\|_1 \leq 1} |f(x)| \leq \sup_{\|x\| \leq 1} |f(x)| \leq 1.$$

This shows that $F \subset B_1(X^*)$.

Now we prove that $(X, \|\cdot\|_1, F)$ is (F^k) uniformly convex for any positive integer k. By Proposition 1 it suffices to prove that $(X, \|\cdot\|_1, F)$ is (F^1) uniformly convex. For the purpose it suffices to prove that for any sequences $\{x_n\}$, and $\{y_n\}$ of $B_1(X)$ with $\|x_n + y_n\|_1 \to 2$, then we have

$$\lim_{n \to \infty} |A(x_n, y_n; f)| = \lim_{n \to \infty} |f(x_n - y_n)| = 0, \qquad \forall f \in F.$$

Otherwise, there exist some $\varepsilon_0 > 0$, $f_{k_0} \in F$ and some subsequences $\{x_{n_m}\} \subset \{x_n\}$, and $\{y_{n_m}\} \subset \{y_n\}$ such that

$$|f_{k_0}(x_{n_m} - y_{n_m})| \geq \varepsilon_0, \qquad m = 1, 2, \dots . \tag{2.10}$$

It follows from $|f_{k_0}(x_{n_m})| \leq 1$, $|f_{k_0}(y_{n_m})| \leq 1$, (2.10), Proposition 3 that there exists $\delta \in (0,1)$ such that

$$\left|\frac{f_{k_0}(x_{n_m}) + f_{k_0}(y_{n_m})}{2}\right|^2 \leq (1-\delta)\frac{|f_{k_0}(x_{n_m})|^2 + |f_{k_0}(y_{n_m})|^2}{2}. \tag{2.11}$$

Therefore we have

$$\frac{\delta}{2^{k_0}}\left|\frac{f_{k_0}(x_{n_m} - y_{n_m})}{2}\right|^2 \leq \frac{\delta}{2^{k_0}}\left(\frac{|f_{k_0}(x_{n_m})| + |f_{k_0}(y_{n_m})|}{2}\right)^2 \leq$$

$$\leq \frac{\delta}{2^{k_0}}\frac{|f_{k_0}(x_{n_m})|^2 + |f_{k_0}(y_{n_m})|^2}{2} \leq$$

$$\leq \frac{1}{2^{k_0}}\left(\frac{|f_{k_0}(x_{n_m})|^2 + |f_{k_0}(y_{n_m})|^2}{2} - \left|\frac{f_{k_0}(x_{n_m}) + f_{k_0}(y_{n_m})}{2}\right|^2\right) \leq$$

$$\leq \frac{\|x_{n_m}\|^2 + \|y_{n_m}\|^2}{2} - \left\|\frac{x_{n_m} + y_{n_m}}{2}\right\|^2 +$$

$$+\sum_{i=1}^{\infty}\frac{1}{2^i}\left(\frac{|f_i(x_{n_m})|^2+|f_i(y_{n_m})|^2}{2}-\left|\frac{f_i(x_{n_m}+y_{n_m})}{2}\right|^2\right)=$$
$$=\frac{1}{2}(\|x_{n_m}\|_1^2+\|y_{n_m}\|_1^2)-\left\|\frac{x_{n_m}+y_{n_m}}{2}\right\|_1^2\leq$$
$$\leq 1-\left\|\frac{x_{n_m}+y_{n_m}}{2}\right\|_1^2\to 0 \qquad (n_m\to\infty).$$

This implies that $|f_{k_0}(x_{n_m}-y_{n_m})|\to 0$ and this leads to a contradiction. Therefore we have

$$\lim_{n\to\infty}|f(x_n-y_n)|=0, \qquad \forall f\in F.$$

This completes the proof. □

3. The F.P.P. for generalized nonexpansive mappings

We are now in a position to prove the main results of this paper.

Theorem 3. *Every Banach space having normal structure has the F.P.P. for generalized nonexpansive mapping.*

Proof. Let K be any nonempty weakly compact convex subset of a given Banach space X, and $T : K\to K$ be any generalized nonexpansive mapping. We define Γ as a collection of all nonempty closed convex subsets E of K with $T(E)\subset E$. It is obvious that $\Gamma\neq\emptyset$ as $K\in\Gamma$. Now we define a partial ordering on Γ by the set inclusion relation. It follows from Smulian Theorem [4, p. 433] and Zorn Lemma that there exists a minimal element $S\in\Gamma$. If S contains only one point, say x, then x is a fixed point of T, and the proof is complete. If S consists of more than one point, then it follows from the normal structure of K, there exists an $x_*\in S$ such that

$$\alpha=\sup_{y\in S}\|x_*-y\|<\operatorname{diam}(S). \tag{3.1}$$

Therefore we have

$$\sup_{y\in S}\|Tx_*-Ty\|\leq\sup_{y\in S}\|x_*-y\|=\alpha.$$

This implies that

$$T(S) \subset B(Tx_*, \alpha) \triangleq \{x \in X : \|Tx_* - x\| \leq \alpha\} . \tag{3.2}$$

Letting $G = S \cap B(Tx_*, \alpha)$ we know that G is a nonempty closed convex set with $T(G) \subset G$. By the minimality of S it gets $G = S$. This implies that $S \subset B(Tx_*, \alpha)$. Therefore we have

$$\sup_{y \in S} \|Tx_* - y\| \leq \alpha. \tag{3.3}$$

Now we consider the set S' defined by

$$S' = \{z \in S : \sup_{y \in S} \|z - y\| \leq \alpha\}. \tag{3.4}$$

It is obvious that S' is a nonempty closed convex subset. We claim that $T(S') \subset S'$. In fact, if $z \in S'$ then $z \in S$ and $Tz \in S$.

Next, it is easy to see that $\overline{\text{co}}T(S) \subset S$, and consequently

$$T(\overline{\text{co}}T(S)) \subset T(S) \subset \overline{\text{co}}T(S) \subset S.$$

This means that $\overline{\text{co}}T(S) \in \Gamma$. In view of the minimality of S we get $S = \overline{\text{co}}T(S)$. Hence $\text{co}T(S)$ is a dense subset of S, and consequently for each $y \in S$ and each $\varepsilon > 0$ there exist $x_i \in S$, $i = 1, 2, \ldots, n$ such that

$$\|y - \sum_{i=1}^{n} \beta_i T x_i\| \leq \varepsilon, \qquad \beta_i \geq 0, \qquad \sum_{i=1}^{n} \beta_i = 1.$$

Hence for each $z \in S'$ we have

$$\|Tz - y\| \leq \|Tz - \sum_{i=1}^{n} \beta_i T x_i\| + \|\sum_{i=1}^{n} \beta_i T x_i - y\| \leq$$

$$\leq \varepsilon + \sum_{i=1}^{n} \beta_i \|Tz - Tx_i\| \leq \varepsilon + \sup_{y \in S} \|z - y\| \leq \varepsilon + \alpha.$$

By the arbitrariness of $y \in S$ and $\varepsilon > 0$ we have $\sup_{y \in S} \|Tz - y\| \leq \alpha$. This means that $Tz \in S'$ i.e. $T(S') \subset S$. Hence $S' = S$. But by the definition of S' it leads

$$\text{diam}(S') = \sup_{x, y \in S'} \|x - y\| \leq \sup_{x \in S', y \in S} \|x - y\| \leq \alpha < \text{diam}(S).$$

Thus we get a contradiction. Therefore S is a singleton.

This completes the proof. □

Remark. The main result of Jaggi [5] is a special case of Theorem 3. In particular, it extends the main results of Browder [2] and Kirk [6].

From Theorem 1, 2 and Theorem 3 we have the following result.

Theorem 4. *Let $(X, \|\cdot\|, F)$ be a Banach space with the S.P. . Then X admits an equivalent norm $\|\cdot\|_1$ defined by (2.1) so that $(X, \|\cdot\|_1)$ has the F.P.P. for generalized nonexpansive mappings.*

As a consequence of Theorem 4 we have the following

Corollary 1. *Let $(X, \|\cdot\|)$ be a separable Banach space. Then X admits an equivalent norm $\|\cdot\|_1$ defined by*

$$\|x\|_1 = (\|x\|^2 + \sum_{i=1}^{\infty} \frac{1}{2^i}|f_i(x)|^2)^{\frac{1}{2}}, \qquad f_i \in F,$$

where F is a set defined by (2.3), so that $(X, \|\cdot\|_1)$ has the F.P.P. for generalized nonexpansive mapping (hence for every kind of mappings which satisfies the condition (1.1), (1.3) or (1.4)).

Remark. This Corollary extends and improves the main results of Dulst [3]. Of course, it also extends the main results of Browder [2] and Kirk [6].

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REZIME

UNIFORMNA KONVEKSNOST I OSOBINA NEPOKRETNE TAČKE

U ovom radu uveden je pojam (F^k) uniformno konveksnih prostora i dokazane su teoreme o nepokretnoj tački. Rezultati koji su dobijeni proširuju neke rezultate Dulsta[3], Jaggia[5], Browdera[2] i Kirka[6].

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